Empirical law of the iterated logarithm for Markov chains with a countable state space

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Abstract

We find conditions which are sufficient and nearly necessary for the compact and bounded law of the iterated logarithm for Markov chains with a countable state space.

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1 Introduction

Let $(S, \mathcal{G}, P)$ be a probability space and let $\mathcal{F}$ be a set of measurable functions on S with an envelope function F finite everywhere. Let $X_1, X_2, ...$ be a strictly stationary sequence of random variables with distribution P.

We say $\mathcal{F}$ satisfies the compact LIL with respect to $\{X_i\}$ if there exists a compact set K in $l^\infty(\mathcal{F})$, where $l^\infty(\mathcal{F})$ is the space of all bounded function on $\mathcal{F}$ with sup norm, such that, with probability one,

$$\left\{ \frac{1}{\sqrt{2n \log \log n}} \sum_{i=1}^{n} (f(X_i) - Ef(X_1)) : f \in \mathcal{F} \right\}_{n=1}^{\infty}$$

is relatively compact and its limit set is K, and $\mathcal{F}$ satisfies the bounded LIL with respect to $\{X_i\}$ if, with probability one,

$$\sup_{n} \sup_{f \in \mathcal{F}} \frac{1}{\sqrt{2n \log \log n}} \left| \sum_{i=1}^{n} (f(X_i) - Ef(X_1)) \right| < \infty.$$

We say the uniform CLT holds for $\mathcal{F}$ if

$$\left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (f(X_i) - Ef(X_1)) : f \in \mathcal{F} \right\}_{n=1}^{\infty}$$

converges weakly, in the space $l^\infty(\mathcal{F})$ to a Gaussian process. If $\mathcal{F}$ is a class of indicator functions on S then the uniform CLT holding for $\mathcal{F}$ implies $\mathcal{F}$ satisfies the compact LIL for i.i.d. sequence of random variables. [7]

In this paper, $\{X_i\}_{i \geq 0}$ is a positive recurrent irreducible Markov chain taking values in countable state space $S = \{1, 2, 3, \cdots\}$ with the unique invariant probability measure π. Let N_i be the i-th hitting time of state 1, and $m_{i,j}$ be the expected minimal number of steps from state i to state j. Levental (1990) [9] proved that

$$\sum_{k=1}^{\infty} \pi(k)\sqrt{m_{1,k}} < \infty \tag{1.1}$$

is a necessary and sufficient condition for the uniform CLT for $\mathcal{F} = \{1_A : A \subset S\}$ for Markov chains satisfying $E(N_2 - N_1)^2 < \infty$. However, the family of indicator functions can be generalized to unbounded classes of functions, $\mathcal{F} = \{f : |f| \leq F\}$, where F is a non-negative function on S, corresponding to the condition [10]

$$\sum_{k=1}^{\infty} F(k)\pi(k)\sqrt{m_{1,k}} < \infty. \tag{1.2}$$

It should also be observed that Levental's result generalizes Durst and Dudley's condition [4]

$$\sum_{k=1}^{\infty} \pi^{\frac{1}{2}}(k) < \infty \tag{1.3}$$

for the uniform CLT for i.i.d. data. That is, since $m_{1,k} = m_{k,k} = (\pi(k))^{-1}$ in the i.i.d. case, (1.1) coincides with (1.3).

We prove the compact LIL and bounded LIL for Markov chains under a weaker condition than the condition (1.2) of the uniform CLT. Suppose $E(N_2 - N_1)^2 < \infty$ and let $\mathcal{F}$ be a compact subset of $L^2(S, \pi)$ with envelope function F satisfying

$$E\left(\sum_{N_1 < j \leq N_2} F(X_j)\right)^2 < \infty.$$

If

$$\frac{1}{\sqrt{\log\log n}} \sum_{k=1}^{n} F(k)\pi(k)\sqrt{m_{1,k}} \to 0 \tag{1.4}$$

as $n \to \infty$ for all orderings of S, (the convergence is relative to the ordering of S) then $\mathcal{F}$ satisfies the compact LIL with respect to $\{X_j\}$.

On the other hand, suppose that $m_{k,1}$ and $F(k)$ are both of polynomial rate, i.e. there are $c > 0$ and $\alpha > 0$ such that

$$\max\{m_{k,1}, F(k)\} \leq ck^{\alpha} \quad \text{for all } k \in S. \tag{1.5}$$

If $\{F1_A : A \subseteq S\}$ satisfies the compact LIL with respect to $\{X_i\}$, then (1.4) holds for all ordering satisfying (1.5).

We also have the bounded LIL result if (1.4) is replaced by

$$\sup_n \frac{1}{\sqrt{\log\log n}} \sum_{k=1}^{n} F(k)\pi(k)\sqrt{m_{1,k}} < \infty. \tag{1.6}$$

In particular, if $\{X_i\}_{i\geq 0}$ is i.i.d., then $m_{k,1}$ is a constant and the condition $E(N_2 - N_1)^2 < \infty$ holds. Thus we have that $\{1_A : A \subseteq S\}$ satisfies the compact LIL (bounded LIL) if and only if

$$(\log\log n)^{-\frac{1}{2}} \sum_{k=1}^{n} \sqrt{\pi(k)} \to 0 \quad (\sup_n (\log\log n)^{-\frac{1}{2}} \sum_{k=1}^{n} \sqrt{\pi(k)} < \infty)$$

for all ordering of S.

Arcones (1995) [1] proved the compact LIL for stationary sequences satisfying absolutely regular mixing conditions under the following two conditions:

$$\sum_{k=1}^{\infty} \beta_k k^{\frac{2}{p-2}} \log\log k < \infty,$$

and the envelope function F satisfies

$$\sum_{k=1}^{\infty} F(k) \left(P(X_1 = k)\right)^{\frac{1}{p}} < \infty,$$

where $p > 2$ and β_k is absolutely regular mixing coefficients. Since a positive recurrent irreducible Markov chain has convergent absolutely regular mixing coefficients [3], one can also obtain LIL results by using empirical process LIL for stationary sequences satisfying absolutely regular mixing conditions. We have example to show that our conditions are less restrictive than those required for a mixing process application to these problems.

2 Statement of the results

Let $\{X_j\}_{j\geq 0}$ be a positive recurrent irreducible Markov chain taking values in $S = \{1, 2, 3, \cdots\}$ with an invariant probability measure π. Let N_i be the i-th hitting time of state 1, i.e.

$$N_1 = \min\{n : n \geq 1, X_n = 1\}$$

and for $i > 1$

$$N_i = \min\{n : n > N_{i-1}, X_n = 1\},$$

and $m_{i,j}$ be the expected minimal number of steps from state i to state j, i.e.

$$m_{i,j} = E\left(\min\{n : n \geq 1, X_n = j\} \mid X_0 = i\right).$$

We have sufficient and nearly necessary conditions for the compact LIL and bounded LIL.

Theorem 1. Suppose

$$E(N_2 - N_1)^2 < \infty, \tag{2.1}$$

and let $\mathcal{F}$ be a compact subset of $L^2(S,\pi)$ with envelope function F satisfying

$$E\left(\sum_{N_1<j\leq N_2} F(X_j)\right)^2 < \infty. \tag{2.2}$$

If

$$\frac{1}{\sqrt{\log\log n}} \sum_{k=1}^{n} F(k)\pi(k)\sqrt{m_{1,k}} \to 0 \tag{2.3}$$

as $n \to \infty$ for all orderings of S, (the convergence is relative to the ordering of S) then $\mathcal{F}$ satisfies the compact LIL with respect to $\{X_j\}$.

On the other hand, suppose that $m_{k,1}$ and $F(k)$ are of polynomial rate, i.e. there are $c > 0$ and $\alpha > 0$ such that

$$\max\{m_{k,1}, F(k)\} \leq ck^{\alpha} \quad \text{for all } k \in S. \tag{2.4}$$

If $\{F1_A : A \subseteq S\}$ satisfies the compact LIL with respect to $\{X_i\}$, then (2.3) holds for all orderings satisfying (2.4).

Theorem 2. Suppose (2.1) and let $\mathcal{F}$ be a compact subset of $L^2(S,\pi)$ with envelope function F satisfying (2.2). If

$$\sup_n \frac{1}{\sqrt{\log\log n}} \sum_{k=1}^{n} F(k)\pi(k)\sqrt{m_{1,k}} < \infty. \tag{2.5}$$

for all orderings of S, (this is relative to the ordering of S) then $\mathcal{F}$ satisfies the bounded LIL with respect to $\{X_i\}$.

On the other hand, suppose that $F(k)$ and $m_{k,1}$ are of polynomial rate, i.e. there are $c > 0$ and $\alpha > 0$ such that (2.4) holds. If $\{F1_A : A \subseteq S\}$ satisfies the bounded LIL with respect to $\{X_i\}$, then (2.5) holds for all orderings satisfying (2.4).

Remarks.
(i) Note that (2.1) and (2.2) are also necessary conditions of the compact LIL or bounded LIL.
(ii) (2.2) implies $F \in L^2(S,\pi)$. Thus $\{F1_A : A \subseteq S\}$ and $\{f : |f| \leq F\}$ are both compact subsets in $L^2(S,\pi)$.
(iii) Define for all $f \in \mathcal{F}$

$$Z_1(f) = \sum_{N_1<i\leq N_2} \left(f(X_i) - \pi(f)\right),$$

then $Z_1(\cdot)$ is almost surely continuous on compact set $\mathcal{F} \subset L^2(S,\pi)$. We consider Z_1 taking value in $\mathcal{C}(\mathcal{F})$, where $\mathcal{C}(\mathcal{F})$ is the class of continuous functions on $\mathcal{F}$. Since $\mathcal{F}$ is compact $\mathcal{C}(\mathcal{F})$ is a separable Banach space. We define $H_{L(Z_1)}$ in

$\mathcal{C}(\mathcal{F})$ by canonical construction [5], and let K be the unit ball of $H_{L(Z_1)}$. The limit set of

$$\left\{\frac{1}{\sqrt{2n\log\log n}}\sum_{i=1}^{n}(f(X_i)-\pi(f)):f\in\mathcal{F}\right\}_{n=1}^{\infty}$$

is $\frac{K}{\sqrt{m_{1,1}}}$.

3

Equivalent condition of the compact LIL and bounded LIL

Denote $a_n=\sqrt{2n\log\log n}$. We define for every $f\in L^1(S,\pi)$ centered sum

$$S_n(f)=\sum_{i=1}^{n}(f(X_i)-\pi(f))$$

and centered sum of blocks

$$Z_j(f)=\sum_{N_j<i\le N_{j+1}}(f(X_i)-\pi(f))$$

for all $j\ge 1$. Then the $Z_j(f)$ are i.i.d.. Let

$$l(n)=\max\{i:N_i\le n\}$$

then

$$S_n(f)=\sum_{1\le i\le N_1}(f(X_i)-\pi(f))+\sum_{j=1}^{l(n)-1}Z_j(f)+\sum_{N_{l(n)}<i\le n}(f(X_i)-\pi(f)).$$

We have

$$\begin{aligned}&\lim_{n\to\infty}\sup_{f\in\mathcal{F}}\frac{1}{\sqrt{n}}\left|\sum_{1\le i\le N_1 \text{ or } N_{l(n)}<i\le n}(f(X_i)-\pi(f))\right|\\&\le\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left|\sum_{1\le i\le N_1 \text{ or } N_{l(n)}<i\le n}(F(X_i)+\pi(F))\right|\\&=0 \text{ a.s.}\end{aligned}$$

from Chung's proof of Theorem 5 on page 106 in [2].(use assumption (2.2)) Thus

$$\frac{1}{a_n}\sup_{f\in\mathcal{F}}\left|S_n(f)-\sum_{j=1}^{l(n)-1}Z_j(f)\right|\to 0 \text{ a.s.}\tag{3.1}$$

as $n \to \infty$. We have the following lemma.

Lemma 1. Suppose $E(N_2 - N_1)^2 < \infty$ and let $\mathcal{F}$ be a compact subset of $L^2(S, \pi)$ with envelope function F satisfying

$$E\left(\sum_{N_1 < j \le N_2} F(X_j)\right)^2 < \infty.$$

Then $\mathcal{F}$ satisfies the compact LIL with respect to $\{X_i\}$ is equivalent to

$$\frac{1}{a_n} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^{n} Z_j(f) \right| \to 0 \quad \text{in probability,} \tag{3.2}$$

and $\mathcal{F}$ satisfies the bounded LIL with respect to $\{X_j\}$ is equivalent to

$$\frac{1}{a_n} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^{n} Z_j(f) \right| \quad \text{is bounded in probability.} \tag{3.3}$$

Proof.

$\mathcal{F}$ satisfies the compact LIL with respect to $\{X_i\}$ is equivalent to that there exist a compact subset K in $l^\infty(\mathcal{F})$ such that

$$\left\{\frac{1}{a_n} S_n(f)\right\}_{f \in \mathcal{F}} \to\to \frac{K}{\sqrt{m_{1,1}}} \quad \text{a.s.} \tag{3.4}$$

(the notation $\{x_n\} \to\to A$ means both $\lim_n d(x_n, A) = 0$ and the cluster set of $\{x_n\}$ is A, for a metric space (X, d) and a sequence $\{x_n\}$ of points in X.)

Using (3.1), (3.4) is equivalent to

$$\left\{\frac{1}{a_n} \sum_{j=1}^{l(n)-1} Z_j(f)\right\}_{f \in \mathcal{F}} \to\to \frac{K}{\sqrt{m_{1,1}}} \quad \text{a.s..}$$

Since $\frac{a_n}{a_{l(n)-1}} \to \sqrt{m_{1,1}}$, where $a_{l(n)-1} = \sqrt{2(l(n)-1) \log\log(l(n)-1)}$, the last expression is equivalent to

$$\left\{\frac{1}{a_{l(n)-1}} \sum_{j=1}^{l(n)-1} Z_j(f)\right\}_{f \in \mathcal{F}} \to\to K \quad \text{a.s.,}$$

and then

$$\left\{\frac{1}{a_n} \sum_{j=1}^{n} Z_j(f)\right\}_{f \in \mathcal{F}} \to\to K \quad \text{a.s.,} \tag{3.5}$$

since $\{l(n) : n = 1, 2, ...\} = \{1, 2, ...\}$.

Since $Z_j(\cdot)$ are almost surely continuous on compact set $\mathcal{F} \subset L^2(S, \pi)$, we can consider Z_j taking value in $\mathcal{C}(\mathcal{F})$, where $\mathcal{C}(\mathcal{F})$ is the class of continuous functions on $\mathcal{F}$ with sup norm. Since $\mathcal{F}$ is compact, $\mathcal{C}(\mathcal{F})$ is a separable Banach space. Thus we can apply those limit theorems in separable Banach spaces.

Note that

$$E\|Z_1\|^2 \leq 2\left[E\left(\sum_{N_1<j\leq N_2} F(X_i)\right)^2 + \pi^2(F)E\left(N_2 - N_1\right)^2\right] < \infty,$$

and $EZ_1 = 0$. By applying Theorem 4.1 in [6] we have that (3.5) is equivalent to

$$\frac{1}{a_n}\left\|\sum_{j=1}^{n} Z_j\right\| \to 0 \quad \text{in probability.}$$

Thus $\mathcal{F}$ satisfies the compact LIL with respect to $\{X_i\}$ is equivalent to (3.2).

Using similar arguments we can obtain that $\mathcal{F}$ satisfies the bounded LIL with respect to $\{X_i\}$ is equivalent to

$$\sup_n \frac{1}{a_n}\left\|\sum_{j=1}^{n} Z_j\right\| < \infty \quad \text{a.s.,} \tag{3.6}$$

and (3.6) is equivalent to

$$\frac{1}{a_n}\sum_{j=1}^{n} Z_j \quad \text{is bounded in probability}$$

by Theorem 4.2 in [6]. ■

4 Proof of sufficient part of the compact LIL

Suppose

$$E(N_2 - N_1)^2 < \infty,$$

$$E\left(\sum_{N_1<j\leq N_2} F(X_j)\right)^2 < \infty$$

and

$$\frac{1}{\sqrt{\log\log n}}\sum_{k=1}^{n} F(k)\pi(k)\sqrt{m_{1,k}} \to 0. \tag{4.1}$$

for all ordering of k in S. By Lemma 1 we only have to show

$$\frac{1}{a_n}\sup_{f\in\mathcal{F}}\left|\sum_{j=1}^{n} Z_j(f)\right| \to 0 \quad \text{in probability.}$$

First we have

$$\sup_{f\in\mathcal{F}}\left|\sum_{j=1}^{n} Z_j(f)\right| \le \sup_{f\in\mathcal{F}}\sum_{k=1}^{\infty}|f(k)|\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| \le \sum_{k=1}^{\infty}F(k)\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right|. \tag{4.2}$$

Thus

$$\begin{aligned} & P\left(\tfrac{1}{a_n}\sup_{f\in\mathcal{F}}\left|\sum_{j=1}^{n} Z_j(f)\right| > \varepsilon\right) \\ \le\ & \underbrace{P\left(\frac{1}{a_n}\sum_{k=1}^{n^2}F(k)\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| > \frac{\varepsilon}{2}\right)}_{I} + \underbrace{P\left(\frac{1}{a_n}\sum_{k=n^2+1}^{\infty}F(k)\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| > \frac{\varepsilon}{2}\right)}_{II}. \end{aligned} \tag{4.3}$$

By Markov's inequality

$$I \le \frac{\sqrt{2}}{\varepsilon}(\log\log\ n)^{-\frac{1}{2}}\sum_{k=1}^{n^2}F(k)E\left(n^{-\frac{1}{2}}\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right|\right).$$

Since $Z_i(\cdot)$ are i.i.d. and centered,

$$E(n^{-\frac{1}{2}}\left|\sum_{i=1}^{n} Z_i(1_{\{k\}})\right|) \le \left(n^{-1}E\left|\sum_{i=1}^{n} Z_i(1_{\{k\}})\right|^2\right)^{\frac{1}{2}} = \left(E\left(Z_1^2(1_{\{k\}})\right)\right)^{\frac{1}{2}}. \tag{4.4}$$

Thus we need to show that

$$(\log\log\ n)^{-\frac{1}{2}}\sum_{k=1}^{n^2}F(k)\left(E\left(Z_1^2(1_{\{k\}})\right)\right)^{\frac{1}{2}} \to 0,$$

and this is equivalent to

$$(\log\log\ n)^{-\frac{1}{2}}\sum_{k=1}^{n}F(k)\left(E\left(Z_1^2(1_{\{k\}})\right)\right)^{\frac{1}{2}} \to 0. \tag{4.5}$$

>¿From Chung [2, p88],

$$E\left(\sum_{N_1<j\le N_2}1_{\{k\}}(X_j)\right)^2 = 2m_{1,1}\pi^2(k)\left(m_{1,k}+m_{k,1}\right) - m_{1,1}\pi(k) \quad \text{for } k \ge 1. \tag{4.6}$$

Thus

$$\left(E\left(Z_1^2(1_{\{k\}})\right)\right)^{\frac{1}{2}} \leq \left[E\left(\sum_{N_1<j\leq N_2} 1_{\{k\}}(X_j)\right)^2\right]^{\frac{1}{2}} \leq \sqrt{2m_{1,1}}\left(\pi(k)\sqrt{m_{1,k}}+\pi(k)\sqrt{m_{k,1}}\right). \tag{4.7}$$

We can obtain the convergence of (4.5) from (4.1) and

$$\frac{1}{\sqrt{\log\log n}}\sum_{k=1}^{n} F(k)\pi(k)\sqrt{m_{k,1}} \to 0. \tag{4.8}$$

for all ordering of k. To show (4.8), from Chung [2, p88],

$$E\left(N_2-N_1\right)^2 = 2m_{1,1}\sum_{k=1}^{\infty}\pi(k)m_{k,1} - m_{1,1}. \tag{4.9}$$

Since $E\left(N_2-N_1\right)^2<\infty$, we thus have

$$\sum_{k=1}^{\infty}\pi(k)m_{k,1}<\infty.$$

Hence

$$\sum_{k=1}^{\infty} F(k)\pi(k)\sqrt{m_{k,1}} \leq \left(\sum_{k=1}^{\infty} F^2(k)\pi(k)\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\pi(k)m_{k,1}\right)^{\frac{1}{2}}<\infty, \tag{4.10}$$

and thus obtain (4.8).

For the other part of (4.3) we have

$$II \leq \frac{2}{\varepsilon a_n}\sum_{k=n^2+1}^{\infty} F(k)E\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| \leq \frac{\sqrt{8}m_{1,1}}{\varepsilon}\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}}\sum_{k=n^2+1}^{\infty} F(k)\pi(k),$$

since

$$E\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| \leq nE\left|Z_1(1_{\{k\}})\right| \leq 2m_{1,1}\pi(k)n.$$

Now we have to show that

$$\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}}\sum_{k=n^2+1}^{\infty} F(k)\pi(k) \to 0.$$

Note

$$\sum_{k=n^2+1}^{\infty} F(k)\pi(k) \leq \left(\sum_{k=n^2+1}^{\infty} F^2(k)\pi(k)\right)^{\frac{1}{2}}\left(\sum_{k=n^2+1}^{\infty}\pi(k)\right)^{\frac{1}{2}} \leq \left(\sum_{k=n^2+1}^{\infty} F^2(k)\pi(k)\right)^{\frac{1}{2}},$$

we will show

$$\frac{n}{\log\log n}\sum_{k=n^2+1}^{\infty}F^2(k)\pi(k)\to 0.$$

Since

$$\sqrt{\pi(k)}=\pi(k)\sqrt{m_{k,k}}\le \pi(k)\sqrt{m_{1,k}}+\pi(k)\sqrt{m_{k,1}},$$

(4.1) and (4.8) imply

$$\frac{1}{\sqrt{\log\log n}}\sum_{k=1}^{n}F(k)\sqrt{\pi(k)}\to 0$$

for all ordering of k. Thus we can assume $F(k)\sqrt{\pi(k)}$ is decreasing, then for n large enough

$$F(n)\sqrt{\pi(n)}\le \frac{1}{n}\sum_{k=1}^{n}F(k)\sqrt{\pi(k)}\le \frac{1}{n}\sqrt{\log\log n}.$$

Thus for n large enough

$$\frac{n}{\log\log n}\sum_{k=n^2+1}^{\infty}F^2(k)\pi(k)\le \frac{n}{\log\log n}\sum_{k=n^2+1}^{\infty}\frac{\log\log k}{k^2},$$

and the right hand side converges to zero as $n\to\infty$. ■

5

Proof of necessary part of the compact LIL

In this section $\mathcal{F}=\{F1_A : A\subseteq S\}$ and $\|\cdot\|$ is the sup norm in $\mathcal{C}(\mathcal{F})$. Suppose

$$E(N_2-N_1)^2<\infty, \tag{5.1}$$

$$E\left(\sum_{N_1<j\le N_2}F(X_j)\right)^2<\infty \tag{5.2}$$

and

$$\frac{1}{a_n}\left\|\sum_{j=1}^{n}Z_j\right\|\to 0 \quad \text{in probability.} \tag{5.3}$$

We need to show that

$$\frac{1}{\sqrt{\log\log n}}\sum_{k=1}^{n}F(k)\pi(k)\sqrt{m_{1,k}}\to 0 \tag{5.4}$$

for all orderings such that $m_{k,1}$ and $F(k)$ are of a polynomial rate, i.e. there are $c > 0$ and $\alpha > 0$ such that

$$\max\{m_{k,1}, F(k)\} \leq ck^{\alpha} \quad \text{for all } k \in S. \tag{5.5}$$

Recall

$$Z_j(f) = \sum_{N_j < i \leq N_{j+1}} (f(X_i) - \pi(f))$$

and define

$$Y_j(f) = \sum_{N_j < i \leq N_{j+1}} f(X_i) - m_{1,1}\pi(f)$$

in this section. To prove (5.4), first we show the following lemma.

Lemma 2. Suppose (5.1), (5.2) and (5.3) hold, then

$$\frac{1}{a_n} \sum_{k=1}^{\infty} F(k) E \left| \sum_{j=1}^{n} Y_j(1_{\{k\}}) \right| \to 0 \quad \text{as } n \to \infty.$$

Proof.

Let

$$U_j(f) = ((N_{j+1} - N_j) - m_{1,1})\, \pi(f),$$

then

$$Z_j(f) = Y_j(f) + U_j(f).$$

Kolmogorov's LIL holds for the i.i.d. sequence $\{N_{j+1} - N_j\}_{j \geq 1}$ since $E(N_2 - N_1)^2 < \infty$, and that is equivalent to

$$\frac{1}{a_n} \left| \sum_{j=1}^{n} ((N_{j+1} - N_j) - m_{1,1}) \right| \to 0 \quad \text{in probability.}$$

We have

$$\frac{1}{a_n} \left\| \sum_{j=1}^{n} U_j \right\| \to 0 \quad \text{in probability,}$$

since $\sup_{f \in \mathcal{F}} \pi(f) = \pi(F) < \infty$. Thus

$$\frac{1}{a_n} \left\| \sum_{j=1}^{n} Y_j \right\| \to 0 \quad \text{in probability.} \tag{5.6}$$

We then claim

$$\frac{1}{a_n} E \left\| \sum_{j=1}^{n} Y_j \right\| \to 0. \tag{5.7}$$

We have the bounded LIL for $\{Y_j\}$ since the bounded LIL holds for $\{Z_j\}$ and $\{U_j\}$, that is

$$\sup_n \frac{1}{a_n}\left\|\sum_{j=1}^{n} Y_j\right\| < \infty \quad \text{a.s.}$$

Then use Corollary 6.12 in [8]

$$E\left[\sup_n \frac{1}{a_n}\left\|\sum_{j=1}^{n} Y_j\right\|\right] < \infty \tag{5.8}$$

is equivalent to

$$E\left[\sup_n \frac{1}{a_n}\|Y_n\|\right] < \infty. \tag{5.9}$$

Since (5.2) holds we have

$$E\left[\sup_n \frac{1}{a_n}\sum_{N_n<i\le N_{n+1}} F(X_i)\right] < \infty.$$

(Here we use that if $\{X_i\}$ is i.i.d. and real valued, then $EX_1^2 < \infty$ implies $E[\sup_n n^{-\frac{1}{2}}|X_n|] < \infty$). Since

$$\|Y_n\| \le \sum_{N_n<i\le N_{n+1}} F(X_i) + m_{1,1}\pi(F),$$

we have (5.9) and thus (5.8) holds. Combine (5.6) and (5.8), we obtain (5.7).

Pointwise, for all ω, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n} Y_j\right\| &= \sup_{f\in\mathcal{F}}\left|\sum_{j=1}^{n} Y_j(f)\right| \\ &= \sup_{f\in\mathcal{F}}\left|\sum_{k=1}^{\infty} f(k)\sum_{j=1}^{n} Y_j(1_{\{k\}})\right| \\ &\ge \sum_{k=1}^{\infty} F(k)\left|1_{\{\sum_{j=1}^{n} Y_j(1_{\{k\}})\ge 0\}}\sum_{j=1}^{n} Y_j(1_{\{k\}})\right|.\end{aligned}$$

Thus

$$E\left\|\sum_{j=1}^{n} Y_j\right\| \ge \frac{1}{2}\sum_{k=1}^{\infty} F(k)E\left|\sum_{j=1}^{n} Y_j(1_{\{k\}})\right|.$$

¿From (5.7) we have

$$\frac{1}{a_n}\sum_{k=1}^{\infty} F(k)E\left|\sum_{j=1}^{n} Y_j(1_{\{k\}})\right| \to 0. \blacksquare$$

Then we need the following lemma.

Lemma 3. There are $c', M > 0$ such that

$$n^{-\frac{1}{2}} E \left| \sum_{j=1}^{n} Y_j(1_{\{k\}}) \right| \geq c' \left[EY_1^2(1_{\{k\}}) \right]^{\frac{1}{2}}$$

for all $n \geq M/\eta_k$, where $\eta_k = P(\tau_k < \tau_1 \mid X_0 = 1)$ and $\tau_i = \min\{n \geq 1 : X_n = i\}$.

Proof. By the Marcinkiewicz-Zygmund inequality

$$n^{-\frac{1}{2}} E \left| \sum_{j=1}^{n} Y_j(1_{\{k\}}) \right| \geq c_1 E \left[\left(n^{-1} \sum_{j=1}^{n} Y_j^2(1_{\{k\}}) \right)^{\frac{1}{2}} \right]$$

where $c_1 > 0$ is a constant which is independent of the random variables. Note that

$$E \left[\left(\frac{1}{n} \sum_{j=1}^{n} Y_j^2(1_{\{k\}}) \right)^{\frac{1}{2}} \right] \geq \left[EY_1^2(1_{\{k\}}) \right]^{\frac{1}{2}} P \left(\frac{1}{n} \sum_{j=1}^{n} Y_j^2(1_{\{k\}}) \geq EY_1^2(1_{\{k\}}) \right).$$

By the Berry-Esseen theorem

$$P \left(\frac{1}{n} \sum_{j=1}^{n} Y_j^2(1_{\{k\}}) \geq EY_1^2(1_{\{k\}}) \right) \geq \frac{1}{2} - \frac{3n^{-\frac{1}{2}} EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}}) \right)^{\frac{3}{2}}}.$$

Let $\tau_i = \min\{n \geq 1 : X_n = i\}$, $\eta_k = P(\tau_k < \tau_1 \mid X_0 = 1)$ and $\lambda_k = P(\tau_1 < \tau_k \mid X_0 = k)$. Thus it is enough to show that there is a $M > 0$ such that

$$\frac{3n^{-\frac{1}{2}} EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}}) \right)^{\frac{3}{2}}} \leq \frac{1}{4} \quad \text{for all } n \geq \frac{M}{\eta_k}. \tag{5.10}$$

Denote $W(k) = \sum_{N_1 < i \leq N_2} 1_{\{k\}}(X_i)$ then

$$Y_1(1_{\{k\}}) = W(k) - EW(k)$$

and

$$\begin{aligned} EW^l(k) &= \sum_{m=1}^{\infty} m^l P(W(k) = m) \qquad (5.11) \\ &= \sum_{m=1}^{\infty} m^l \left[\eta_k (1 - \lambda_k)^{m-1} \lambda_k \right] \\ &= \eta_k (1 - \lambda_k)^{-1} \lambda_k \sum_{m=1}^{\infty} m^l (1 - \lambda_k)^m. \end{aligned}$$

Thus by computation $EW(k) = \frac{\eta_k}{\lambda_k}$,

$$\begin{aligned}
&EY_1^6(1_{\{k\}}) \\
&\leq EW^6(k) \\
&= \frac{\eta_k}{\lambda_k^6}\left(1 + 57(1-\lambda_k) + 302(1-\lambda_k)^2 + 302(1-\lambda_k)^3 + 57(1-\lambda_k)^4 + (1-\lambda_k)^5\right) \\
&\leq \frac{720\eta_k}{\lambda_k^6}
\end{aligned}$$

and

$$\begin{aligned}
&EY_1^4(1_{\{k\}}) \\
&= E\left(W(k) - \frac{\eta_k}{\lambda_k}\right)^4 \\
&= \frac{\eta_k}{\lambda_k^4}\left(\begin{array}{c} 1 + 11(1-\lambda_k) + 11(1-\lambda_k)^2 + (1-\lambda_k)^3 - 4\eta_k(1-\lambda_k)^2 \\ -16\eta_k(1-\lambda_k) - 4\eta_k + 6\eta_k^2(1-\lambda_k) + 6\eta_k^2 - 3\eta_k^3 \end{array}\right).
\end{aligned}$$

Since

$$\sum_{k=1}^{\infty}\eta_k \leq \sum_{k=1}^{\infty}\frac{\eta_k}{\lambda_k} = \sum_{k=1}^{\infty}EW(k) = \sum_{k=1}^{\infty}m_{1,1}\pi(k) = m_{1,1} < \infty,$$

most η_k are small. Thus

$$EY_1^4(1_{\{k\}}) \geq \frac{\eta_k}{2\lambda_k^4}$$

except for finitely many k. Hence there is a $M_1 > 0$ such that

$$\frac{EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}})\right)^{\frac{3}{2}}} \leq M_1\eta_k^{-\frac{1}{2}}.$$

Thus

$$\frac{3n^{-\frac{1}{2}}EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}})\right)^{\frac{3}{2}}} \leq \frac{1}{4} \quad \text{for all } n \geq \frac{144M_1}{\eta_k}. \blacksquare$$

Proof of (5.4).

Since

$$\begin{aligned}
\left(EY_1^2(1_{\{k\}})\right)^{\frac{1}{2}} &\geq \left(E\left(\sum_{N_1<i\leq N_2}1_{\{k\}}(X_i)\right)^2\right)^{\frac{1}{2}} - m_{1,1}\pi(k) \\
&\geq \sqrt{m_{1,1}}\pi(k)m_{1,k}^{\frac{1}{2}} - 2m_{1,1}\pi(k),
\end{aligned}$$

from (4.6), it's enough to show

$$(\log\log n)^{-\frac{1}{2}} \sum_{k=1}^{n} F(k) \left(EY_1^2(1_{\{k\}})\right)^{\frac{1}{2}} \to 0. \tag{5.12}$$

Note that use (5.11)

$$EY_1^2(1_{\{k\}}) = \eta_k \left(\frac{2-\lambda_k-\eta_k}{\lambda_k^2}\right) \le \frac{2\eta_k}{\lambda_k^2} \tag{5.13}$$

and

$$\begin{aligned} m_{k,1} &\ge \sum_{j=1}^{\infty} j \left(P\left(\tau_1 > \tau_k \mid X_0 = k\right)\right)^{j-1} P\left(\tau_1 < \tau_k \mid X_0 = k\right) \qquad (5.14) \\ &= \sum_{j=1}^{\infty} j(1-\lambda_k)^{j-1}\lambda_k \\ &= \lambda_k^{-1}. \end{aligned}$$

Let

$$U = \left\{k : F(k) \left(EY_1^2(1_{\{k\}})\right)^{\frac{1}{2}} \ge k^{-2}\right\}.$$

Use (5.13), (5.14) and assumption (5.5) for all $k \in U$

$$\eta_k \ge \frac{1}{2}\lambda_k^2 EY_1^2(1_{\{k\}}) \ge \frac{1}{2}\frac{1}{m_{k,1}^2}\frac{k^{-4}}{F^2(k)} \ge \frac{k^{-(4\alpha+4)}}{2c^4}.$$

>From lemma 2 and lemma 3 we have

$$(\log\log n)^{-\frac{1}{2}} \sum_{k:\eta_k \ge \frac{M}{n}} F(k) \left[EY_1^2(1_{\{k\}})\right]^{\frac{1}{2}} \to 0. \tag{5.15}$$

Note

$$k \le \left(\frac{n}{2Mc^4}\right)^{\frac{1}{4\alpha+4}}$$

implies

$$\frac{k^{-(4\alpha+4)}}{2c^4} \ge \frac{M}{n},$$

thus (5.15) implies

$$(\log\log n)^{-\frac{1}{2}} \sum_{k\in U \text{ and } k \le \left(\frac{n}{2Mc^4}\right)^{\frac{1}{4\alpha+4}}} F(k) \left[EY_1^2(1_{\{k\}})\right]^{\frac{1}{2}} \to 0.$$

Since $\sum_{k\notin U} F(k)(EY_1^2(1_{\{k\}}))^{\frac{1}{2}} \le \sum_{k\notin U} k^{-2} < \infty$,

$$(\log\log n)^{-\frac{1}{2}} \sum_{k=1}^{\left[\left(\frac{n}{2Mc^4}\right)^{\frac{1}{4\alpha+4}}\right]} F(k) \left(EY_1^2(1_{\{k\}})\right)^{\frac{1}{2}} \to 0.$$

and this is equivalent to (5.12). ■

6 Proof of the bounded LIL

In view of (3.3) in Lemma 1, the proof of the bounded LIL now follows as indicated in Sections 4 and 5 for the compact LIL. Here, of course, to show (3.3), one uses (2.5) rather than (2.3) at various places. The details are straight forward.

7 Comparison with mixing results

Arcones (1995) [1] proved the compact LIL for stationary sequences satisfying absolutely regular mixing conditions under the following two conditions:

$$\sum_{k=1}^{\infty} \beta_k k^{\frac{2}{p-2}} \log\log k < \infty, \tag{7.1}$$

and the envelope function F satisfies

$$\sum_{k=1}^{\infty} F(k) \left(P(X_1 = k)\right)^{\frac{1}{p}} < \infty, \tag{7.2}$$

where $p > 2$ and β_k is absolutely regular mixing coefficients.

To show our conditions are less restrictive than Arcones' conditions for Markov chains, we present an example such that the conditions $E(N_2 - N_1)^2 < \infty$,

$$E\left(\sum_{N_1 < j \leq N_2} F(X_j)\right)^2 < \infty \tag{7.3}$$

and

$$\sum_{k=1}^{\infty} F(k) \sqrt{E\left(\sum_{N_1 < j \leq N_2} 1_{\{k\}}(X_j)\right)^2} < \infty \tag{7.4}$$

hold (condition (7.4) implies (2.3)), but Arcones' conditions (7.1) and (7.2) fail.

Example. Let $\{X_i\}$ be a stationary Markov chain with transition probability

$$P_{n,n+1} = \left(\frac{n}{n+1}\right)^s, \; P_{n,1} = 1 - P_{n,n+1} \quad \text{for all } n \geq 1 \text{ and some } s > 1.$$

Let $F(k) = k^t$ for some $t \geq 0$. We choose s and t such that (7.3) and (7.4) hold, but (7.1) and (7.2) can not both hold.

From example 3.2 in [10] $\pi(k) \approx k^{-s}$, $P\left(N_2 - N_1 = n\right) \approx n^{-s-1}$,

$$E\left(\sum_{N_1 < j \leq N_2} 1_{\{k\}}(X_j)\right)^2 \approx k^{-s}$$

and $\beta_k \gg k^{1-s}$. Thus

$$\sum_{k=1}^{\infty} F(k)(\pi(k))^{\frac{1}{p}} \approx \sum_{k=1}^{\infty} k^{t-\frac{s}{p}} < \infty \quad \text{only if} \quad p \le \frac{s}{t+1},$$

$$E\left(\sum_{N_1<j\le N_2} F(X_j)\right)^2 = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} k^t\right)^2 P\left(N_2-N_1=n\right) \approx \sum_{n=1}^{\infty} n^{2t-s+1}$$

and

$$\sum_{k=1}^{\infty} F(k)\sqrt{E\left(\sum_{N_1<j\le N_2} 1_{\{k\}}(X_j)\right)^2} \approx \sum_{k=1}^{\infty} k^{t-\frac{s}{2}}.$$

Hence (7.3) and (7.4) hold if $s > 2t+2$. We take $s = 4.2$ and $t = 1$. If (7.2) holds, then $p \le \frac{21}{10}$. But

$$\sum_{k=1}^{\infty} \beta_k k^{\frac{2}{p-2}} \log\log k \gg \sum_{k=1}^{\infty} k^{-3.2} k^{20} \log\log k = \infty,$$

thus (7.1) fails. ■

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