The CLT for Markov chains with a countable state space embedded in the space l_p

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2 June, 2000

Abstract

We find necessary and sufficient conditions for the CLT for Markov chains with a countable state space embedded in the space l_p for $p \geq 1$. This result is an extension of the uniform CLT over the family of indicator functions in Levental (Stochastic Processes Appl. 34 (1990)245-253), where the result is equivalent to our case $p = 1$. A similar extension for the uniform CLT over a family of possibly unbounded functions in Tsai(Taiwan. J. Math. 1(4)(1997)481-498) is also obtained.

1 Introduction and results

Let $\{X_i\}_{i\geq 0}$ be a Markov chain with state space $\mathbf{N} = \{1, 2, 3, \cdots\}$. We assume that the chain is irreducible and positive recurrent. It is well known that $\{X_i\}$ has a unique invariant probability measure, which we denote by π. Let

$$N_1 = \min\{n \geq 1 : X_n = 1\}$$

and

$$N_i = \min\{n > N_{i-1} : X_n = 1\}$$

for $i \geq 2$. This random variable is called the i-th entrance time of the state 1. Let

$$m_{i,j} = E(\min\{n : n \geq 1, X_n = j \mid X_0 = i\})$$

denote the mean entrance time to state j for a Markov chain starting at state i.

Let $f \in L_1(\mathbf{N}, \pi)$. We denote the centered sum by

$$S_n(f) = \sum_{i=1}^{n} (f(X_i) - \pi(f)).$$

It is well known that $S_n(f)/\sqrt{n}$ converges in law to a normal distribution if f is bounded and the Markov chain satisfies the condition (see section 16 in Chung (1967))

$$E(N_2 - N_1)^2 < \infty. \tag{1.1}$$

Levental (1990) extended this CLT to a uniform CLT. That is, we say that the CLT holds uniformly over $\mathcal{F} \subset L_2(\mathbf{N}, \pi)$ if the process

$$\left\{S_n(f)/\sqrt{n}\right\}_{f \in \mathcal{F}}$$

converges weakly to a Gaussian process indexed by $\mathcal{F}$ whose sample paths are uniformly continuous with respect to the L_2-norm metric of $L_2(\mathbf{N}, \pi)$. By assuming that the Markov chain satisfies condition (1.1), Levental proved that the CLT holds uniformly over $\mathcal{F} = \{1_A : A \subseteq \mathbf{N}\}$ if and only if

$$\sum_{k=1}^{\infty} \pi(k) m_{1,k}^{\frac{1}{2}} < \infty. \tag{1.2}$$

Levental's result is a generalization of a result by Durst and Dudley (1981) for i.i.d. observations, where the condition is

$$\sum_{k=1}^{\infty} \pi^{\frac{1}{2}}(k) < \infty. \tag{1.3}$$

Of course, in the i.i.d. case $m_{1,k} = (\pi(k))^{-1}$, so (1.2) coincides with (1.3).

The uniform CLT with a countable space $\mathbf{N}$ can be formulated as results in the space l_p. Here and throughout this paper, l_p denotes the space of real functions on $\mathbf{N}$ for which

$$\|f\|_p = \left(\sum_i |f(i)|^p\right)^{\frac{1}{p}} < \infty,$$

($\sup_i |f(i)| < \infty$ if $p = \infty$) for all $0 < p \leq \infty$. Let B_p be the unit ball of l_p. Let $\mathbf{N}$ be embedded into the canonical basis of l_p, that is, define $\xi : \mathbf{N} \to l_p$ by $\xi(k) = e_k$, where

$$e_k = (0, 0, ..., 0, 1^{k\text{-th}}, 0, ...) \in l_1, \ \ k \in \mathbf{N}$$

is the canonical basis of l_p. We say $\xi(X_i)$ satisfies the CLT in the space l_p and denote by

$$\xi(X_i) \in CLT(l_p),$$

if and only if the normalized centered sum

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\xi(X_i) - \xi(\pi))$$

converges weakly to a Gaussian measure in the space l_p. Note that

$$f(X_i) = \langle f, \xi(X_i) \rangle ,$$

for all f. Thus $\xi(X_i) \in CLT(l_1)$ if and only if the CLT holds uniformly over B_∞, or, what is equivalent, over $\{1_A : A \subseteq \mathbf{N}\}$. In other words, in Levental (1990), the uniform CLT is equivalent to $\xi(X_i) \in CLT(l_1)$.

Our purpose is to find the condition for $\xi(X_i) \in CLT(l_p)$ for $p > 1$, which is equivalent to that the CLT holds uniformly over B_q, where $q = p/(p-1)$ is the conjugate exponent to p. We obtain necessary and sufficient conditions for $\xi(X_i) \in CLT(l_p)$, $1 \le p < \infty$, in the following theorem.

Theorem 1. For $1 \le p \le 2$, $\xi(X_i) \in CLT(l_p)$ if and only if (1.1) holds and

$$\sum_{k=1}^{\infty} \pi^p(k) m_{1,k}^{\frac{p}{2}} < \infty. \tag{1.4}$$

For $2 < p < \infty$, $\xi(X_i) \in CLT(l_p)$ if and only if (1.1) holds.

Note that condition (1.1) actually implies (1.4) for $p > 2$.

The proof of the result uses first blocking between the successive N_i to reduce the problem to independent random variables, and then applies the CLT for independent random variables taking values in l_p; see Jain (1977) and Pisier and Zinn (1978).

The families of functions over which the CLT for Markov chains holds uniformly need not be bounded. In [7] we give a necessary and sufficient condition for the CLT to hold uniformly over collections of functions of the form $\mathcal{F} = \{f : |f| \le F\}$, where F is a positive function in $L_1(S, \pi)$. This condition is

$$\sum_{k=1}^{\infty} F(k)\pi(k) m_{1,k}^{\frac{1}{2}} < \infty.$$

We derive a similar result in the space l_p that is relative to the uniform CLT over $\mathcal{F} = \{f : |f| \le F\}$. Define $\xi_F : \mathbf{N} \to l_p$ by

$$\xi_F(k) = F(k) e_k.$$

Then

$$f(X_i) = \langle f, \xi(X_i) \rangle = \left\langle \frac{f}{F}, \xi_F(X_i) \right\rangle ,$$

for all $f \in \{f : |f| \le F\}$. Thus $\xi_F(X_i) \in CLT(l_1)$ if and only if the CLT holds uniformly over $\mathcal{F} = \{f : |f| \le F\}$. We extend the result to $p > 1$ as follows.

Theorem 2. For $1 \le p \le 2$, $\xi_F(X_i) \in CLT(l_p)$ if and only if (1.1) holds and

$$\sum_{k=1}^{\infty} F^p\,(k)\,\pi^p(k) m_{1,k}^{\frac{p}{2}} < \infty. \tag{1.5}$$

For $2 < p < \infty$, $\xi_F(X_i) \in CLT(l_p)$ if and only if (1.1) and (1.5) hold and

$$t^2 P\left(\left\|\sum_{N_1 < i \leq N_2} \xi_F(X_i)\right\|_p > t\right) \to 0 \; as \; t \to \infty. \tag{1.6}$$

2 Proof

We will use the CLT for independent random variables taking values in l_p; see Corollary 1 in Jain (1977) and Theorem 5.1 in Pisier and Zinn (1978).

Let $\{Y_i\}$ be i.i.d. random variables in the space l_p.

Theorem A. (Jain, 1977)

For $1 \leq p \leq 2$, $Y_i \in CLT(l_p)$ if and only if

$$\sum_{k=1}^{\infty} \left(E\left[Y_1^{(k)}\right]^2\right)^{\frac{p}{2}} < \infty, \tag{2.1}$$

where $Y_1^{(k)}$ is the k-th component of Y_1.

Theorem B. (Pisier and Zinn, 1978)

For $2 < p < \infty$, $Y_i \in CLT(l_p)$ if and only if (2.1) holds and

$$t^2 P(\|Y_1\|_p > t) \to 0 \text{ as } t \to \infty. \tag{2.2}$$

In particular, if $E\,\|Y_1\|_p^2 < \infty$ then (2.1) and (2.2) hold for $2 < p < \infty$.

Let $\{X_i\}_{i\geq 0}$ be a positive recurrent and irreducible Markov chain taking values in $\mathbf{N}$ with the unique invariant probability measure π. Denote by N_i the i-th entrance time of state 1. Let

$$l(n) = \max\{i : N_i \leq n\}$$

be the number of times state 1 is visited. We also need the following lemma; see Tsai (1997).

Lemma C. The random variable $(l(n) - \frac{n}{m_{1,1}})/\sqrt{n}$ converges in law to a normal distribution if and only if (1.1) holds.

The centered sum S_n can be divided into four parts

$$\sum_{1 \leq i \leq N_1} (\xi(X_i) - \xi(\pi)) + \sum_{k=1}^{\left[\frac{n}{m_{1,1}}\right]} Z_k + \left(\sum_{k=1}^{l(n)-1} Z_k - \sum_{k=1}^{\left[\frac{n}{m_{1,1}}\right]} Z_k\right) + \sum_{N_{l(n)} < i \leq n} (\xi(X_i) - \xi(\pi)),$$

where the blocks $\{Z_k\}$,

$$Z_k = \sum_{N_k < i \le N_{k+1}} (\xi(X_i) - \xi(\pi)),$$

are i.i.d.. Since the first term does not depend on n

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left\| \sum_{1\le i\le N_1} (\xi(X_i) - \xi(\pi)) \right\|_p \to 0 \quad a.s. \tag{2.3}$$

for all $p \ge 1$. Regarding the fourth part, it is bounded in probability, (see Theorem 8 in section 14 in Chung), thus

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left\| \sum_{N_{l(n)} < i \le n} (\xi(X_i) - \xi(\pi)) \right\|_p \to 0 \text{ in probability} \tag{2.4}$$

for all $p \ge 1$. For the third part we need the following two propositions.

Proposition 1. Let $Z_1, Z_2, ...$ be i.i.d. random variables taking values in the space l_p with $EZ_1 = 0$ and $E\|Z_1\|_p < \infty$ for some $p \ge 1$. Assume that (1.1) holds. Then

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left\| \sum_{k=1}^{l(n)-1} Z_k - \sum_{k=1}^{\left[\frac{n}{m_{1,1}}\right]} Z_k \right\|_p \to 0 \quad \text{in probability.} \tag{2.5}$$

Proof. Fix $\varepsilon > 0$ and claim that

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \left\| \sum_{k=1}^{l(n)-1} Z_k - \sum_{k=1}^{\left[\frac{n}{m_{1,1}}\right]} Z_k \right\|_p > \varepsilon \right) = 0.$$

Given $\delta > 0$, by Lemma C there is a $c > 0$ such that

$$P\left(\frac{n}{m_{1,1}} - c\sqrt{n} \le l(n) - 1 \le \frac{n}{m_{1,1}} + c\sqrt{n} \right) \ge 1 - \delta.$$

Observe that, on $\{\frac{n}{m_{1,1}} - c\sqrt{n} \le l(n) - 1 \le \frac{n}{m_{1,1}} + c\sqrt{n}\}$,

$$\left\| \sum_{k=1}^{l(n)-1} Z_k - \sum_{k=1}^{\left[\frac{n}{m_{1,1}}\right]} Z_k \right\|_p \le \max_{1\le i\le c\sqrt{n}} \left\| \sum_{k=1}^{i} Z_k \right\|_p.$$

Then, taking $m = [c\sqrt{n}]$,

$$P\left(\frac{1}{\sqrt{n}}\left\|\sum_{k=1}^{l(n)-1} Z_k - \sum_{k=1}^{\left[\frac{n}{m_{1,1}}\right]} Z_k\right\|_p > \varepsilon\right)$$

$$\leq \delta + P\left(\frac{1}{\sqrt{n}}\max_{1\leq i\leq c\sqrt{n}}\left\|\sum_{k=1}^{i} Z_k\right\|_p > \varepsilon\right)$$

$$\leq \delta + P\left(\frac{1}{m}\max_{1\leq i\leq m}\left\|\sum_{k=1}^{i} Z_k\right\|_p > \frac{\varepsilon}{c}\right).$$

Since $EZ_1 = 0$ and $E\left\|Z_1\right\|_p < \infty$, the strong law of large numbers in the space l_p (see Corollary 7.10 in Ledoux and Talagrand (1991)) gives

$$\frac{1}{m}\left\|\sum_{k=1}^{m} Z_k\right\|_p \to 0 \quad a.s. \text{ as } m \to \infty,$$

thus

$$\frac{1}{m}\max_{1\leq i\leq m}\left\|\sum_{k=1}^{i} Z_k\right\|_p \to 0 \quad a.s. \text{ as } m \to \infty.$$

Consequently,

$$P\left(\frac{1}{m}\max_{1\leq i\leq m}\left\|\sum_{k=1}^{i} Z_k\right\|_p > \frac{\varepsilon}{c}\right) \to 0 \text{ as } m \to \infty.$$

Since δ is arbitrary we have

$$P\left(\frac{1}{\sqrt{n}}\left\|\sum_{k=1}^{l(n)-1} Z_k - \sum_{k=1}^{\left[\frac{n}{m_{1,1}}\right]} Z_k\right\|_p > \varepsilon\right) \to 0.$$

Proposition 2. If (1.1) holds then $E[(\|Z_1\|_p)^2] < \infty$ for all $p \geq 1$.

Proof. Let card$\{A\}$ denote the cardinality of A. We have

$$\begin{aligned}
&\left\|\sum_{N_1<i\le N_2}\xi(X_i)\right\|_p\\
&=\left(\sum_{k=1}^{\infty}(\mathrm{card}\{i: X_i=k, N_1<i\le N_2\})^p\right)^{\frac{1}{p}}\\
&\le\left(\left(\sum_{k=1}^{\infty}\mathrm{card}\{i: X_i=k, N_1<i\le N_2\}\right)^p\right)^{\frac{1}{p}}\\
&=N_2-N_1.
\end{aligned}$$

Thus

$$E\left[(\|Z_1\|_p)^2\right]\le E(N_2-N_1)^2<\infty.$$

Proof of Theorem 1. First assume that (1.1) holds. For all $p\ge 1$, we obtain, by using (2.3), (2.4) and (2.5),

$$\left\|\frac{1}{\sqrt{n}}\left(S_n-\sum_{k=1}^{\left[\frac{n}{m_{1,1}}\right]}Z_k\right)\right\|_p\to 0 \textit{ in probability.}$$

Thus $\xi(X_i)\in CLT(l_p)$ if and only if $Z_i\in CLT(l_p)$.

For $1\le p\le 2$, by Theorem A, $Z_i\in CLT(l_p)$ if and only if

$$\sum_{k=1}^{\infty}\left(E\left[Z_1^{(k)}\right]^2\right)^{\frac{p}{2}}<\infty. \tag{2.6}$$

Note that

$$Z_1^{(k)}=\sum_{N_1<i\le N_2}\left(1_{\{X_i=k\}}-\pi(k)\right).$$

The second moment of a block (Theorem 7) in section 14 in Chung gives

$$E\left[Z_1^{(k)}\right]^2=2m_{1,1}\pi^2(k)\left(m_{1,k}+m_{k,1}\right)-m_{1,1}\pi(k)-m_{1,1}^2\pi^2(k),$$

and

$$\sum_{k=1}^{\infty}\pi(k)m_{k,1}<\infty. \tag{2.7}$$

By (2.7)

$$2m_{k,1}\le\frac{1}{\pi(k)}=m_{k,k}\le m_{1,k}+m_{k,1},$$

and

$$m_{1,k} \le m_{1,k} + m_{k,1} \le 2m_{1,k},$$

except for finitely many k. Hence the condition (2.6) holds if and only if the condition (1.4) holds. Therefore we have that $\xi(X_i) \in CLT(l_p)$ if and only if (1.4) holds for $1 \le p \le 2$.

For $2 < p < \infty$, we have $E[(\|Z_1\|_p)^2] < \infty$ by Proposition 2. By Theorem B, we obtain that $Z_i \in CLT(l_p)$, which is proved to be equivalent to $\xi(X_i) \in CLT(l_p)$. Note that $E[(\|Z_1\|_p)^2] < \infty$ implies (2.6), which is proved to be equivalent to the condition (1.4). Thus (1.1) implies (1.4) for $2 < p < \infty$.

It remains to prove that if $\xi(X_i) \in CLT(l_p)$ for some $1 \le p < \infty$ then (1.1) holds. Note that $\xi(X_i) \in CLT(l_p)$ implies

$$\frac{\sum_{i=1}^n 1_{\{1\}}(X_i) - n\pi(1)}{\sqrt{n}} \text{ converges in law to } \left\langle 1_{\{1\}}, G \right\rangle.$$

Since $\sum_{i=1}^n 1_{\{1\}}(X_i) = l(n)$ and $\pi(1) = \frac{1}{m_{1,1}}$, the random variable

$$\frac{1}{\sqrt{n}}(l(n) - \frac{n}{m_{1,1}})$$

converges in law to a normal distribution $\left\langle 1_{\{1\}}, G \right\rangle$. Hence we have (1.1) by Lemma C.

Proof of Theorem 2. Assume that (1.1) holds. Let

$$Z_k^F = \sum_{N_k < i \le N_{k+1}} \left(\xi_F(X_i) - \xi_F(\pi)\right).$$

Then the assumption of Proposition 1 holds, that is we have

$$E\left\|Z_1^F\right\|_p \le E\left\|Z_1^F\right\|_1 \le E\left(\sum_{N_1 < i \le N_2} F(X_i)\right) = E(N_2 - N_1)\pi(F) < \infty,$$

since, by assumption, F is a positive function in $L_1(\mathbf{N}, \pi)$. As in the proof of Theorem 1, we can obtain that $\xi_F(X_i) \in CLT(l_p)$ if and only if $Z_k^F \in CLT(l_p)$ for all $p \ge 1$.

For $1 \le p \le 2$, we have that $Z_k^F \in CLT(l_p)$ if and only if

$$\sum_{k=1}^{\infty} \left(E\left[\left(Z_1^F\right)^{(k)}\right]^2\right)^{\frac{p}{2}} < \infty, \tag{2.8}$$

by Theorem A. Note that

$$\left(E\left[\left(Z_1^F\right)^{(k)}\right]^2\right)^{\frac{p}{2}} = F^p(k)\left(2m_{1,1}\pi^2(k)\left(m_{1,k} + m_{k,1}\right) - m_{1,1}\pi(k) - m_{1,1}^2\pi^2(k)\right)^{\frac{p}{2}},$$

and thus (2.8) holds if and only if the condition (1.5) holds. Therefore we have that $\xi_F(X_i) \in CLT(l_p)$ if and only if (1.5) holds.

For $2 < p < \infty$, $Z_k^F \in CLT(l_p)$ if and only if (2.8) holds and

$$t^2 P(\left\|Z_1^F\right\|_p > t) \to 0 \ as \ t \to \infty, \tag{2.9}$$

by Theorem B. Note that (2.9) is equivalent to (1.6). Thus we have that $\xi_F(X_i) \in CLT(l_p)$ if and only if (1.5) and (1.6) hold.

The proof of the necessity of the condition (1.1) is the same as the proof of Theorem 1.

Acknowledgments:

The author thanks Professors Evarist Giné and Jim Kuelbs for pointing out that empirical results on a countable space can be formulated as results in the space l_p. He is also indebted to the referees and the Associate Editor for their careful reading and helpful suggestions.

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