Stochastic Calculus — Final Examination Solutions

June 17, 2005

There are 12 problems and 10 points each.

1. (Property of Brownian Bridge)
   Let $B_t^0 = \{B_t, 0 \le t \le 1 | B_1 = 0\}$ be a Brownian bridge, and define $dX_t = -\frac{X_t}{1-t}dt + dB_t$, with $X_0 = 0$.
   a) Show that $\{B_t^0, 0 \le t \le 1\}$ and $\{X_t, 0 \le t \le 1\}$ have the same distribution.
   b) Show that $Y_t = (1+t)B^0_{\frac{t}{1+t}}$ is a standard Brownian motion.
   sol:
   (a)
   $(1)B_t^0 = B_t - tB_1, \because B_t \sim N(0,t), B_1 \sim N(0,1), tB_1 \sim N(0,t^2)$ $\therefore$ linear combination of normal distribution is still normal. Compute mean and variance.
   $EB_t^0 = 0,$
   $EB_s^0 B_t^0 = s(1-t) \text{if} s < t \Rightarrow \text{Var} B_t^0 = t(1-t)$
   $B_t^0 \sim N(0,\ t(1-t))\ \forall 0 \le t \le 1.$
   (2) According to the solution of OU process, $X_t = (1-t)\int_0^t \frac{1}{1-s}dB_s$
   X_t is the linear combination of N.D., therefore it is still a N.D. Compute mean and variance
   $X_t = 0,$
   If$s \le t : EX_sX_t$
   $= (1-s)(1-t)E[\int_0^s \frac{1}{1-u}dB_u \int_0^t \frac{1}{1-v}dB_v]$
   $= (1-s)(1-t)E[\int_0^s \frac{1}{1-u}dB_u(\int_0^s \frac{1}{1-v}dB_v + \int_s^t \frac{1}{1-v}dB_v)]$
   $= (1-s)(1-t)E[(\int_0^s \frac{1}{1-u}dB_u)^2]$ (independent increment)
   $= (1-s)(1-t)E[\int_0^s (\frac{1}{1-u})^2 ds]$ (Ito's isometry)
   $= (1-s)(1-t)\frac{s}{1-s} = s(1-t) \Rightarrow \text{Var}X_t = t(1-t)$
   $X_t \sim N(0,\ t(1-t))$
   (b)
   $\because B_{\frac{t}{1+t}} \sim N(0,\ \frac{t}{1+t}), B_{\frac{s}{1+s}} \sim N(0,\ \frac{s}{1+s}), B_1 \sim N(0,\ 1) \therefore Y_t \sim$N.D.
   $EX_t = E(1+t)(B_{\frac{t}{1+t}} - \frac{t}{1+t}B_1) = 0$

If $0 \leq s < t \leq 1$ $(0 \leq \frac{s}{1+s} < \frac{t}{1+t} \leq 1)$:
$EX_sX_t = (1+s)(1+t)[EB_{\frac{s}{1+s}}B_{\frac{t}{1+t}} - \frac{t}{1+t}EB_{\frac{s}{1+s}}B_1 - \frac{s}{1+s}EB_{\frac{t}{1+t}}B_1 + \frac{s}{1+s}\frac{t}{1+t}EB_1^2] = s \Rightarrow$
$\text{Var}Y_t = t$
$Y_t \sim N(0,t)$
$(1)Y_0 = 0$
(2)Independent increment
$\forall 0 = t_0 < t_1 < ... < t_n = 1 : Y_{t_1} - Y_{t_0}, Y_{t_2} - Y_{t_1}, ..., Y_{t_n} - Y_{t_{n-1}}$
$\text{Cov}(Y_{t_i} - Y_{t_{i-1}}, Y_{t_j} - Y_{t_{j-1}}) = 0$
(3)stationary increment
$E(Y_t - Y_s) = 0$
$\text{Var}(Y_t - Y_s) = \text{Var}(Y_t) + \text{Var}(Y_s) - 2\text{Cov}(Y_t, Y_s) = t + s - 2s = t - s$
$Y_t - Y_s \sim N(0, t-s)$
$(4)Y_t \sim N(0,t)$
$(5)B_t$ is continuous in t $\Rightarrow Y_t$ is also continuous in t
From (1)$\sim$(5), Y_t is a SBM.

2. a) Solve the following stochastic differential equation

$$dX_t = (-\alpha X_t + \beta)dt + \sigma dB_t, \quad \text{where } X_0 = x_0 \text{ and } \alpha > 0.$$

b) Verify the solution can be written as

$$X_t = e^{-\alpha t}(x_0 + \frac{\beta}{\alpha}(e^{\alpha t} - 1) + \sigma \int_0^t e^{\alpha s} dB_s).$$

c) Show that X_t converges in distribution, as $t \to \infty$, and find the limiting distribution.
d) Find the covariance $\text{Cov}(X_s, X_t)$ for $s < t$.
sol:
(a)
$dX_t + \alpha X_t dt = \beta dt + \sigma dB_t$
multiply $e^{\alpha t}$ for both sides
$d(e^{\alpha t}X_t) = \beta E^{\alpha t}dt + \sigma e^{\alpha t}dB_t$
$\Rightarrow e^{\alpha s}X_s|_0^t = \beta \int_0^t e^{\alpha s}ds + \sigma \int_0^t e^{\alpha s}dBs$

$\Rightarrow X_t = e^{-\alpha t}(x_0 + \frac{\beta}{\alpha}(e^{\alpha t} - 1) + \sigma \int_0^t e^{\alpha s} dB_s)$

(b)

$\frac{\partial X_t}{\partial t} = -\alpha X_t + \beta$

$\frac{\partial X_t}{\partial B_t} = \sigma$

$\frac{\partial^2 X_t}{\partial B_t^2} = \sigma$

By Ito's formula:

$dX_t = (-\alpha X_t + \beta)dt + \sigma dB_t$

(c)

(1)$EX_t = x_0 e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t}) \to \frac{\beta}{\alpha}$ as $t \to \infty$

(2)$\text{Var}X_t = \sigma^2 E \int_0^t e^{-2\alpha(t-s)} ds = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \to \frac{\sigma^2}{2\alpha}$ as $t \to \infty$

Therefore $X_t \xrightarrow{\mathcal{D}} N(\frac{\beta}{\alpha}, \frac{\sigma^2}{2\alpha})$ as $t \to \infty$

(d) $\text{Cov}(X_s, X_t) = \sigma^2 e^{-\alpha(t+s)} \text{Cov}(\int_0^s e^{\alpha u} dB_u, \int_0^t e^{\alpha u} dB_u)$

$= \sigma^2 e^{-\alpha(t+s)} E \int_0^{s \wedge t} e^{2\alpha u} du$

$= \frac{\sigma^2}{2\alpha} e^{-\alpha(t+s)} (e^{2\alpha(s \wedge t)} - 1)$

3. a) Give a precise description of the simplest Girsanov theorem.

b) Let L be the line given by the equation $y = a + bt$ with $a > 0$. Define

$$\tau_L = \inf\{t : B_t = a + bt\}.$$

Use the simplest Girsanov theorem (Brownian motion with drift) to derive the probability density function $f_{\tau_L}(t)$ of τ_L.

sol:

(a)

If the process $\{B_t\}$ is a P-BM and Q is the measure on $C[0,T]$ induced by the process $X_t = B_t + \mu t$, then every bounded Borel measurable function W on the space $C[0,T]$ satisfies

$$E_Q(W) = E_P(W M_T),$$

where M_t is the P-mtgle defined by $M_t = \exp(\mu B_t - \frac{\mu^2 t}{2})$

(b)

$\tau_L = \inf\{t : B_t = a + bt\}, \tau_a = \inf\{t : X_t = a\},$ where$X_t = B_t - bt$

$P(\tau_L \leq t) = Q(\tau_a \leq t) = E_Q(1(\tau_a \leq t)) = E_P(1(\tau_a \leq t)M_T)$(simplest Girsanov theorem)

where$M_t = \exp\{-bB_T - \frac{b^2}{2}T\}$

$= E_P(1(\tau_a \leq t)M_{t\wedge\tau_a})$ ($\{\tau_a \leq t\}$ is $\mathcal{F}_{t\wedge\tau_a}$measurable)

$= E_P(1(\tau_a \leq t)\exp(-ab - \frac{b^2}{2}\tau_a))$

$= \int_0^t \exp(-ab - \frac{b^2}{2}s)\frac{a}{s^{\frac{3}{2}}}\phi(\frac{a}{\sqrt{s}})ds$

$f_{\tau_a}(t) = \frac{\partial}{\partial t}P(\tau_L \leq t) = \frac{a}{t^{3/2}}\phi(\frac{a+bt}{\sqrt{t}})$ for $t \geq 0$.

4. a) State the Martingale representation theorem.

   b) State the Feynman-Kac representation theorem for Brownian motion.

   Sol:

   (a) X_t is an $\{\mathcal{F}_t\}$-martingale, where $\{\mathcal{F}_t\}$ is the standard Brownian filtration, if $\exists a$ and T, such that $E(X_T^2) < \infty$,
   and $\exists!$ $\phi(s,\omega) \in \mathcal{H}^2[0,T]$, such that $X_t = \displaystyle\int_0^t \phi(s,\omega)dB_s$, $\forall 0 \leq t \leq T$.

   (b) Suppose that the function $q : R \to R$ is bounded. Consider

$$\begin{cases} u_t(t,x) & = \frac{1}{2}u_{xx}(t,x) + q(x)u(t,x) \\ u(0,x) & = f(x) \end{cases}$$

   where $f : R \to R$ is also bounded.
   If $u(t,x)$ is the unique bounded solution of the function, then

$$u(t,x) = E[f(x + B_t)exp(\int_0^t q(x + B_s)ds)]$$

5. The Black-Scholes model is assumed to be

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dB_t, \\ d\beta_t &= r\beta_t dt. \end{aligned}$$

   a) By using arbitrage theory to show the procedure of deriving the Black-Scholes PDE for European call option.

b) Show how to transform this PDE into a heat equation.

6. (continued)

c) Use Fourier transform technique to derive the solution of the above heat equation.

d) Derive Black-Scholes formula for European call option via Feynman-Kac representation.

Sol:

(a) Hedging price $Y_t = \beta_t B_t + \gamma_t S_t$.

Ito's formula:

$$dY = (\frac{\partial Y}{\partial t} + \mu S \frac{\partial Y}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2})dt + \sigma S \frac{\partial Y}{\partial S} dW_t$$

And

$$\begin{aligned} dY &= \beta_t dB_t + \gamma_t dS_t \\ &= (r\beta_t B_t + \mu\gamma_t S_t)dt + \sigma\gamma_t S_t dW_t \end{aligned}$$

Matching coefficient:

$$\left\{ \begin{aligned} \gamma_t &= \frac{\partial Y}{\partial S} \\ \beta_t &= \frac{1}{rB_t}(\frac{\partial Y}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2}) \end{aligned} \right.$$

$\therefore Y_t = \frac{1}{rB_t}(\frac{\partial Y}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2})B_t + \frac{\partial Y}{\partial S} S_t$

$\Rightarrow \left\{ \begin{array}{l} \frac{\partial Y}{\partial t}(T, S_t) + \frac{1}{2}\sigma^2(T, S_t) S_T^2 \frac{\partial^2 Y}{\partial S^2}(T, S_t) = 0 \\ Y(T, S) = f(T, S) = (S - K)^+ \end{array} \right.$

(b) Let

$$\left\{ \begin{array}{l} \theta = \sigma^2(T - t) \\ \\ z = \ln S + (r - \frac{\sigma^2}{2})(T - t) \end{array} \right.$$

Set $u(\theta, z) = e^{r(T-t)Y(t,S_t)}$, then

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial \theta} = \frac{1}{2}\frac{\partial^2 u}{\partial z^2} \\ u(0, z) = f(z) = (e^z - K)^+ \end{array} \right.$$

(c) Take Fourier transform:

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}(w,\theta) = \frac{1}{2}(iw)^2\hat{u}(w,\theta) = -\frac{1}{2}\hat{u}(w,\theta) \\ \hat{u}(0,w) = \hat{f}(w) \end{cases}$$

$\Rightarrow \hat{u}(\theta,w) = \hat{f}(w)e^{-\frac{1}{2}w^2\theta}$
Inverse Fourier transform:

$$u(\theta,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{-\frac{1}{2}w^2\theta}e^{iwz}dw$$

By transform of Gaussian, we have $\mathcal{F}(e^{-\frac{z^2}{2\theta}})(w) = \sqrt{\theta}e^{-\frac{1}{2}w^2\theta}$
$\Rightarrow e^{-\frac{z^2}{2\theta}} = \mathcal{F}(\frac{1}{\sqrt{\theta}}e^{-\frac{z^2}{2\theta}})(w) \equiv \mathcal{F}(g_\theta(z))(w)$
Therefore,

$$\begin{aligned} u(\theta,z) &= f * g_\theta(z) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)g_\theta(z-y)dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)\frac{1}{\sqrt{\theta}}e^{-\frac{(z-y)^2}{2\theta}}dy \\ &= \frac{1}{\sqrt{2\pi\theta}}\int_{-\infty}^{\infty}f(y)e^{-\frac{(z-y)^2}{2\theta}}dy \\ &= \frac{1}{\sqrt{2\pi\theta}}\int_{-\infty}^{\infty}(e^y-K)^+e^{-\frac{(z-y)^2}{2\theta}}dy \end{aligned}$$

(d) $y = W_{theta} + z \quad \Rightarrow \quad dy = dW_\theta$, where $\{W_\theta\}$ is the standard Brownian motion.

$$\begin{aligned} E(e^{W_\theta+z}-K)^+ &= E(e^z e^{\sqrt{\theta}W_1}-K)^+ \\ &= E(e^{z+\frac{\theta}{2}}e^{\sqrt{\theta}W_1-\frac{\theta}{2}}-K)^+ \\ &= e^{z+\frac{\theta}{2}}\Phi(\frac{z-\ln K+\theta}{\sqrt{\theta}}) - Ke^{z+\frac{\theta}{2}}\Phi(\frac{z-\ln K}{\sqrt{\theta}}) \\ &= S_0\Phi(\frac{\ln(\frac{S_0}{K})+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}) - Ke^{-rT}\Phi(\frac{\ln(\frac{S_0}{K})+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}) \end{aligned}$$

7. Derive the pricing formula for European put option by using the technique of change of numeraire and Girsanov theorem.
payoff $X = (K - S(T))^+$
$p = \tilde{E}[e^{-rT}(K-S(T))^+]$
$= \tilde{E}[e^{-rT}(K-S(T))I(k \geq S(T)]$
$= e^{-rT}K\tilde{P}(K \geq S(T)) - e^{-rT}\tilde{E}[S(T)I(K \geq S(T))]$
$= (1) - (2)$

$K \geq S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z} \Leftrightarrow z \leq -\frac{1}{\sigma\sqrt{T}}[\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T] \equiv -d_2,$
where $d_2 = \frac{1}{\sigma\sqrt{T}}[\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T]$
$(1) = Ke^{-rT}\tilde{P}(z \leq -d_2) = Ke^{-rT}\Phi(-d_2)$
$(2) = S_0\tilde{E}[e^{-rT}\frac{S(T)}{S(0)}I(K \geq S(T))]$
(Let $\frac{d\bar{P}}{d\tilde{P}} = e^{-rT}\frac{S(T)}{S(0)} = e^{\sigma\tilde{W}_T - \frac{\sigma^2}{2}T}$
Let $\bar{W}_t = \tilde{W}_t - \sigma t$
$k \geq S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z} = S_0 e^{(r+\frac{\sigma^2}{2})T+\sigma\bar{W}_T}$
$\Rightarrow z \leq -d_1)$
$= S_0\bar{P}(z \leq -d_1) = S_0\Phi(-d_1)$ Therefore, $p = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$

8. Barrier option pricing:
Find the arbitrage price of a contingent claim that pays M if the stock price gets to a level K or higher during the time period $[0, T]$ and that pays zero otherwise. In other words, find a formula for the arbitrage price of the claim

$$X = MI_{\{\sup\limits_{t\in[0,T]} S_t > K\}}.$$

sol:
$\sup\limits_{t\in[0,T]} S_t > K \Leftrightarrow \sup\limits_{t\in[0,T]} B_t + \frac{1}{\sigma}(r - \frac{\sigma^2}{2})t \geq \frac{1}{\sigma}\ln\frac{K}{S_0}$
Let $a = \frac{1}{\sigma}\ln\frac{K}{S_0}, b = -\frac{1}{\sigma}(r - \frac{\sigma^2}{2})t$
$\tau = \inf\{B_t = a + bt\}$
$f_\tau = \frac{a}{\sqrt{2\pi t^3}}e^{-\frac{(a+bt)^2}{2t}}$
$u_0 = e^{-rT}\tilde{E}[M1(\sup\limits_{t\in[0,T]} S_t > K)] = e^{-rT}M\tilde{P}(\sup\limits_{t\in[0,T]} S_t > K)$
$= e^{-rT}MP(\tau \leq T)$ $(\text{Law}(\mu + \sigma B_t, t \leq T|\tilde{P}_T) = \text{Law}(r + \sigma B_t, t \leq T|P_T))$
$= e^{-rT}MP(\tau_L \leq T)$
$= e^{-rT}M[1 - \Phi(\frac{\ln\frac{K}{S_0} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{\tau}}) + \frac{K}{S_0}e^{\frac{2}{\sigma^2}(r-\frac{\sigma^2}{2})}\Phi(\frac{-\ln\frac{K}{S_0} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})]$

9. a) State the Poisson process martingale property theorem.
b) Let $N(t)$ be a Poisson process with intensity λ. Show that

$$\exp\{\ln(1-u)N(t) + u\lambda t\}, \quad 0 < u < 1,$$

is a martingale.

Sol:

(a) Let $N(t)$ be a Poisson process with intensity λ. Then $M(t) = N(t) - \lambda t$ is a martingale.

(b)

$$\begin{aligned}
E[\exp\{\ln(1-u)N(t+s)\}|\mathcal{F}_t] &= E[\exp\{\ln(1-u)N(t) + \ln(1-u)(N(t+s) - N(t))\}|\mathcal{F}_t] \\
&= \exp\{\ln(1-u)N(t)\}E[\exp\{\ln(1-u)(N(t+s) - N(t))\}|\mathcal{F}_t] \\
&= \exp\{\ln(1-u)N(t)\}E[\exp\{\ln(1-u)(N(t+s) - N(t))\}] \\
&= \exp\{\ln(1-u)N(t)\}\exp\{-u\lambda s\}
\end{aligned}$$

Therefore, $E[\exp\{\ln(1-u)N(t+s) + u\lambda(t+s)\}|\mathcal{F}_t] = \exp\{\ln(1-u)N(t) + u\lambda t\}$

10. Consider the stock and bond model given by

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dB_t \quad \text{and} \quad d\beta_t = r(t, S_t)\beta_t dt,$$

where all of the model coefficients $\mu(t, S_t), \sigma(t, S_t)$, and $r(t, S_t)$ are given by explicit functions of the current time and the current stock price. Use the coefficient matching method to show that arbitrage price at time t of a European option with terminal time T and payout $h(S_T)$ is given by $f(t, S_t)$, where $f(t, S_t)$ is the solution of the terminal value problem

$$\begin{aligned}
f_t(t, S_t) &= -\frac{1}{2}\sigma^2(t,x)f_{xx}(t,x) - r(t,x)xf_x(t,x) + r(t,x)f(t,x), \\
f(T, x) &= h(x).
\end{aligned}$$

Sol:

(a) For European call option, the payout is $h(S_t) = (S_T - K)^+$.

Consider the replicating portfolio at time t, V_t

Let a_t: the number of units of stock.

b_t: the number of units of the bond.

Total value of the portfolio at time t is

$$V_t = a_t S_t + b_t \beta_t$$

We require that the restructuing of the portfolio be self-financing.

So, the requirement is

$$dV_t = a_t dS_t + b_t d\beta_t$$

Hence put (1) and (2) into (4)

$$\begin{aligned} dV_t &= \mu dt + \sigma(t, S_t)dB_t \\ &= [a_t u(t, S_t) + b_t r(t, S_t)\beta_t]dt + a_t\sigma(t, S_t)dB_t \end{aligned}$$

$\because V_t = f(t, S_t)$ and Ito formula for geometric Brownian Motion. $\Rightarrow$

$$\begin{aligned} dV_t &= f_t(t, S_t)dt + (1/2)f_{xx}(t, S_t)dS_t dS_t + f_x(t, S_t)dS_t \\ &= [f_t(t, S_t) + (1/2)f_{xx}(t, S_t\sigma^2)(t, S_t) + f_x(t, S_t\mu(t, S_t))]dt + f_x(t, S_t)\sigma(t, S_t)dB_t \end{aligned}$$

Compare (1) and (2) $\Rightarrow$

$$\begin{aligned} a_t &= f_x(t, S_t) \\ b_t &= \tfrac{1}{r(t,S_t)\beta_t}(f_t(t, S_t) + 1/2 f_{xx}(t, S_t)\sigma^2(t, S_t)) \\ \therefore V_t &= f(t, S_t) \\ &= a_t S_t + b_t\beta_t \\ &= f_x(t, S_t)S_t + \tfrac{1}{r(t,S_t)\beta_t}(f_t(t, S_t) + 1/2 f_{xx}(t, S_t)\sigma^2(t, S_t)) \end{aligned}$$

The arbitrage price at time t of European option with terminal time T is $f(t, x)$ which is the solution of the terminal value problem.

$$\begin{cases} f_t(t, x) = -\frac{1}{2}\sigma^2(t, x)f_{xx}(t, x) - r(t, x)xf_x(t, x) + r(t, x)f(t, x) \\ f(T, x) = h(x) \end{cases}$$

(b)from (a)

$$\begin{aligned} a_t &= f_x(t, S_t) \\ b_t &= \tfrac{1}{r(t,S_t)\beta_t}(f_t(t, S_t) + 1/2 f_{xx}(t, S_t)\sigma^2(t, S_t)) \\ V_t &= f(t, S_t) = a_t S_t + b_t\beta_t \end{aligned}$$

Hence they replicate $h(S_T)$.

11. The price of a security U is defined as the product of two asset prices R and S, i.e., $U_t = R_t S_t$. Suppose R and S are Ito processes given by

$$\begin{aligned} dR_t &= 2(100 - R_t)dt + S_t dB_t, \\ dS_t &= R_t S_t dt + S_t dB_t. \end{aligned}$$

Show that $\{tU_t, t \geq 0\}$ is also an Ito process, and find its expected instantaneous rate of return at time 0 if $R_0 = 0.15$ and $S_0 = 35$. Sol:

(a).

$$\begin{aligned} dU_t &= U_R dR + U_S dS + U_{RS} dR dS \\ &= S_t(2(100 - R_t)dt + S_t dB_t) + R_t(R_t S_t dt + S_t dB_t) + S_t^2 dt \\ &= (200S_t - 2S_t R_t + R_t^2 S_t + S_t^2)dt + (S_t^2 + R_t S_t)dB_t \end{aligned}$$

Let $f = tU_t$, by Ito formula

$$\begin{aligned} dtU_t &= f_t dt + f_U dU + \tfrac{1}{2} f_{UU}(dU)^2 \\ &= U_t dt + t dU_t \\ &= U_t dt + (200tS_t - 2tU_t + tU_t R_t + tS_t^2)dt + (tS_t^2 + tR_t S_t)dB_t \end{aligned}$$

(b).

$$\begin{aligned} E(\tfrac{dtU_t}{dt}) &= U_t dt + 200tS_t - 2tU_t + tU_t R_t + tS_t^2 \\ &= U_0 + 0 \quad (t \to \infty) \\ &= R_0 S_0 \\ &= 0.15 \cdot 35 \\ &= 5.25 \end{aligned}$$

12. Assume that X follows a geometric Brownian motion with drift α and volatility σ. The economy is risk-neutral, and the risk-free rate of interest is r. A machine prints a certificate worth $X(t)$ at random time t generated by a Poisson arrival process with intensity λ. What is the value of the machine? Hint: By how much should the asset value change at the time the certificate is printed? Assume that V is linear in X.

sol:

$dX = \alpha X dt + \sigma X dW$ To find the value V of the machine, which pays a cash flow of $X(t)$ determined by the Poisson process:

$V = V(X)$

By Ito's lemma:

$dV = V_X dX + \frac{1}{2} V_{XX}(dX)^2$

$= [\alpha X V_X + \frac{1}{2}\sigma^2 X^2 V_{XX}]dt + \sigma X V_X dW$

Expected cash gain (ECG) $= E[dV] = [\alpha X V_X \frac{1}{2}\sigma^2 X^2 V_{XX}]dt$

Expected cash flow (ECF) $= X\lambda dt$ (a cash flow paid with probability λdt)

Total Revenue (TR)=ECG+ECF=$[\alpha X V_X + \frac{1}{2}\sigma^2 X^2 V_{XX} + X\lambda]dt$

Arbitrage theory:

$[\alpha X V_X + \frac{1}{2}\sigma^2 X^2 V_{XX} + X\lambda]dt = rVdt$

$\Rightarrow \alpha X V_X + \frac{1}{2}\sigma^2 X^2 V_{XX} + X\lambda = rV$

Let $V = AX + B, V_X = A, V_{XX} = 0$

$r(AX + B) = \alpha XA + 0 + X\lambda$

$\Rightarrow rA = \alpha A + \lambda rB = 0$

$A = \frac{\lambda}{r-\alpha}, B = 0$

Hence, $V = \frac{\lambda X}{r-\alpha}$