Financial Time Series

Topic 7: Modelling Return Distributions

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OUTLINE

1. Empirical Finding on return distributions
2. Models for Return Distributions
3. Determining the Tail Shape of a Return Distribution
4. Testing for Covariance Stationarity
5. Modelling for Central Part of Return Distributions
Empirical finding on return distributions

- Characteristics: Fat tails, Excess kurtosis, Skewness
- Three return series:
  - daily returns of London FT30 from 1935 - 1994
  - daily returns of the S&P500
  - daily dollar/sterling exchange rate
- Descriptive statistics confirm the preceding stylized facts (Table 5.1).
- Graphical displays: smoothed histogram; Q-Q plots.
Table 5.1. *Descriptive statistics on returns distributions*

<table>
<thead>
<tr>
<th></th>
<th>$T$</th>
<th>mean</th>
<th>median</th>
<th>std.dev</th>
<th>max</th>
<th>min</th>
<th>range</th>
<th>skew</th>
<th>kurt</th>
</tr>
</thead>
<tbody>
<tr>
<td>FT30</td>
<td>15003</td>
<td>0.022</td>
<td>0.000</td>
<td>1.004</td>
<td>10.78</td>
<td>-12.40</td>
<td>23.1</td>
<td>-0.14</td>
<td>14.53</td>
</tr>
<tr>
<td>Dollar</td>
<td>5192</td>
<td>-0.008</td>
<td>0.000</td>
<td>0.647</td>
<td>4.67</td>
<td>-3.87</td>
<td>13.2</td>
<td>-0.70</td>
<td>6.51</td>
</tr>
</tbody>
</table>
Smoothed Histogram

• Consider $P(-h/2 < X \leq h/2) = \int_{-h/2}^{h/2} f(x)\,dx$ so that $P(-h/2 \leq X < h/2) \approx f(\xi)h$. Also, $P(-h/2 \leq X < h/2) \approx n^{-1}\#\{X_i \in [-h/2, h/2]\}$.

• We have the density estimate

\[ \hat{f}_h(x) = (nh)^{-1} \sum_{i=1}^{n} I(X_i \in B_{x,h}) \]

• Consider the kernel density estimate

\[ \hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right) \]

where $K$ is a kernel function with the following properties

- $K$ is symmetric around 0 and integrate to 1.
- If $K$ is a pdf, so it the kernel estimate.
- Kernel estimates is location invariant.

Quantile-Quantile Plots

• If $X$ has a strictly increasing distribution function, $F$, the $p$th quantile of the distribution is the value of $x_p$ such that $F(x_p) = p$ or $x_p = F^{-1}(p)$.

• In a Q-Q plot, the quantiles of one distribution are plotted against those of the other.

• Suppose $G(y) = F(y - h)$. Then

\[ y_p = x_p + h \]
and a Q-Q plot would be a straight line with slope 1 and intercept $h$.

- Let $r_1, \ldots, r_n$ be the returns in the sample period. The order statistics of the sample are these values arranged in increasing order: $r_{(1)}$ being the sample minimum and $r_{(n)}$ the maximum.

- For $i = np$ and $f(x_p) \neq 0$
  \[ \sqrt{n}(r_{(i)} - x_p) \to N(0, p(1 - p)/f^2(x_p)) \].

- Plot $x_p = \frac{i}{n}$ of a normal distribution against the order statistics of sample returns. If the sample returns are normally distributed, the plot should be close to a straight line.
Models for Return Distributions

Stable Distribution

- Mandelbrot (1963) proposed the use of stable distributions to model the fat-tailed and highly peaked features of return distributions.
- The stable distribution is a generalization of normal ones in that they are stable under addition.
- Recall that the characteristic function of a random variable with distribution $F$ is
  \[ \varphi(t) = \int_{-\infty}^{\infty} \exp(itx) dF(x). \]
- The stable characteristic function is given by
  \[ \varphi(t) = \exp(-\sigma^\alpha |t|^\alpha), \]
  where $0 < \alpha \leq 2$ is the characteristic exponent, $\sigma$ is a scale parameter.
- Note that if $X_i$ are i.i.d. with characteristic function $\varphi(t)$, then $S_n = \frac{X_1 + \cdots + X_n}{n^{1/\alpha}}$ has characteristic function
  \[ \left[ \varphi \left( \frac{t}{n^{1/\alpha}} \right) \right]^n = \varphi(t). \]
- Because $S_n$ and $X_i$ has the same distribution, this shows that the stable distribution indeed is stable under addition.
• If daily returns follow a stable distribution, then weakly, monthly and quarterly returns can be viewed as the sum of daily returns, they too will follow stable distributions having identical characteristic exponents.

• The existing literature also deals with the correlation of stable process over time. The moving average of correlated stable random variables are also stable as long as certain conditions on the coefficients are satisfied.

• The inverse Fourier transformation gives the density function of stable distribution

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\sigma^\alpha |t|^\alpha) \exp(-itx)dt
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} \exp(-\sigma^\alpha |t|^\alpha) \cos(tx)dt. \quad (1)
\]

• Normal distribution corresponds to \( \alpha = 2 \). For \( \alpha < 2 \), all moments greater than \( \alpha \) are infinite.

• Regular variation at infinity and the tail index \( \alpha \):

\[
\lim_{s \to \infty} \frac{1 - F(sx)}{1 - F(s)} = x^{-\alpha}.
\]

• The stable distribution displays a power declining tail, \( x^{-\alpha} \), rather than an exponential decline as is the case with the normal. Thus it has fatter tails than normal ones.
Volatility clustering

• Consider $ARCH(1)$ with normal innovations

\[ X_t = U_t \sigma_t \]

where $U_t \sim NID(0, 1)$ and $\sigma_t^2 = w + \beta X_{t-1}^2$.

• From the two preceding equations, we have

\[ X_t^2 = wU_t^2 + \beta U_t^2 X_{t-1}^2 = B_t + A_t X_{t-1}^2. \tag{2} \]

• $ARCH(1)$ process may also exhibit fat tails. De Haan et al. (1989) show that the $X_t$ of (2) regularly varies at infinity and has a tail index $\zeta$ defined explicitly by the equation

\[ \Gamma \left( \frac{\zeta + 1}{2} \right) = \pi^{1/2} (2\beta)^{-\zeta/2}, \]

where $\Gamma$ is the gamma function as long as $\beta < \bar{\beta} = 2^\nu \approx 3.56856$, where $\nu$ is the Euler’s number.

• It can be shown that $\zeta = 2$ at $\beta = 1$, $\zeta = \infty$ at $\beta = 0$, and $\zeta = 0.00279$ at $\beta = \bar{\beta}$.

• It thus follows that, in terms of tail behavior, the stable and ARCH models partially overlap.

• At $\beta = 0$ and 1, the two models have normal tails, while for $1 < \beta < \bar{\beta}$ the tail indices can be equal. For $0 < \beta < 1$, $\zeta > 2$, $X_t$ is covariance stationary and there is no stable counterpart, whereas for $\zeta < 0.00279$ there is no ARCH counterpart.
Determining the Tail Shape

- For stable distribution with \( \alpha < 2 \), the tails are a function of \( \alpha \) and display a power decline

\[
P(X > x) = P(X < -x) = C^\alpha x^{-\alpha}.
\]

- The tail index \( \zeta \) may be defined for distribution other than the stable ones. As we have seen for the GARCH process, the tail index will not equal the characteristic exponent, although it will determine the maximal finite exponent, i.e. the tail index is such that \( E|X^k| < \infty \) for all \( 0 \leq k < \zeta \).

- If \( \zeta < 2 \) then the variance of \( X \) is infinite and \( X \) may be characterized as being generated by a stable distribution for which \( \alpha = \zeta \).

- If \( \zeta \geq 2 \), the variance of \( X \) is finite but the distribution is not necessarily normal and may thus still have fat tails. For example, it may be Student-\( t \), in which case \( \zeta \) defines the degrees of freedom.

- Distributions such as normal and the power exponential possess all moments with infinite \( \zeta \) so that they may be described as thin tailed.

- Loretan and Philips (1994) formalize this by defining the tail behavior of the distribution of \( X \) to take the form
\[ P(X > x) = C \zeta x^{-\zeta}[1 + \zeta_R(x)], \quad x > 0 \]
\[ P(X < -x) = C \zeta x^{-\zeta}[1 + \zeta_L(x)], \quad x > 0, \]

where \( \zeta_i \to 0 (i = R, L) \) as \( x \to \infty \).

- The parameter \( C \) and \( \zeta \) can be estimated using order statistics. If \( X_1, \cdots, X_T \) are the order statistics, then \( \zeta \) can be estimated by

\[
\hat{\zeta} = \left[ s^{-1} \sum_{j=1}^{s} \log X_{(T-j+1)} - \log X_{(T-s)} \right]^{-1}.
\]

An estimate of the scale dispersion is

\[
\hat{C} = (s/T)X_{(T-s)}^{\hat{\zeta}}.
\]

- The preceding estimates depend on the truncation parameter \( s \). Suggestions on how to find \( s \) are as follows.

  - Try different values of \( s \) and selecting an \( s \) in the region over which \( \hat{\zeta} \) is more or less a constant.
  - Use simulation to choose \( s \) such that MSE of \( \hat{\zeta} \) is minimized.
  - \( s \) not exceeding \( 0.1T \).
  - Use the asymptotic theory of Hall (1982), from which the MSE of the limit distribution of \( \hat{\zeta} \) is minimized by choosing \( s(T) = [\lambda T^{2/3}] \), where \( \lambda \) is estimated adaptively by

\[
\hat{\lambda} = |\hat{\zeta}_1 2^{1/2}(T/s_2)(\hat{\zeta}_1 - \hat{\zeta}_2)|^{2/3}.
\]
Here $\hat{\zeta}_1$ and $\hat{\zeta}_2$ are preliminary estimates of $\zeta$ using the data truncation $s_1 = [T^\sigma]$ and $s_2 = [T^\tau]$, respectively, where $0 < \sigma < 2/3 < \tau < 1$.

- Confidence intervals and hypothesis testing for $\zeta$ and $C$ can be calculated using results from Hall such that

$$s^{1/2}(\hat{\zeta} - \zeta) \sim N(0, \zeta^2)$$

and

$$s^{1/2}(\log T/s)^{-1}(\hat{C}_s - C) \sim N(0, C^2).$$

- Given an estimate of the tail index $\zeta$, extreme return levels that are rare exceeded can be established by extrapolating the empirical distribution, and this can be useful for analyzing ‘safety first’ portfolio selection strategies.

- A consistent estimate of the ‘excess level’ $\hat{x}_p$, for which

$$P(\max_{1 \leq i \leq k} X_i \leq \hat{x}_p) = 1 - p$$

for small $p$ and given $k$ is given by

$$\hat{x}_p = \frac{(kr/pT)^{\hat{\gamma}}}{1 - 2^{-\hat{\gamma}}}[X(T-r) - X(T-2r)] + X(T-r),$$

where $\hat{\gamma} = \hat{\zeta}^{-1}$, $r = s/2$, and $k$ is the time period considered.
Empirical Evidence on Tail Indices

- The exchange rate returns are fat tailed with $\zeta < 4$, and during a variety of fixed exchange rate regimes, tail indices are in the region $1 \leq \zeta \leq 2$. For floating regimes, however, $\zeta$ tends to exceed 2.

- The interpretation is that a float lets exchange rates adjust more smoothly than regimes that involve some amount of fixity.

- Empirical studies on US stock and bond market returns yield tail index estimates in the region $2 < \zeta < 4$, so that although the distribution are fat tailed, they appear to be characterized by finite variances.

- Estimates of tail indices for our three series are shown in Table 5.2. All return distributions have estimated tail indices lying in the region $2 < \zeta < 4$ with the left tail indices usually being a little smaller than the right.

- Figure 5.2 plots the left tail shapes of the empirical DF of the returns in $\log_{10}[P(X < -x)]$ scale against $\log_{10} x$ for $x > 0$. 

<table>
<thead>
<tr>
<th>s</th>
<th>Left tail</th>
<th>Right tail</th>
<th>Both tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>3.167 (0.633)</td>
<td>3.598 (0.720)</td>
<td>4.377 (0.875)</td>
</tr>
<tr>
<td>50</td>
<td>3.138 (0.444)</td>
<td>2.847 (0.403)</td>
<td>3.253 (0.460)</td>
</tr>
<tr>
<td>75</td>
<td>3.135 (0.362)</td>
<td>3.028 (0.350)</td>
<td>3.357 (0.385)</td>
</tr>
<tr>
<td>100</td>
<td>3.305 (0.330)</td>
<td>3.113 (0.311)</td>
<td>3.082 (0.308)</td>
</tr>
<tr>
<td>320</td>
<td>2.937 (0.164)</td>
<td>2.922 (0.163)</td>
<td>3.111 (0.174)</td>
</tr>
<tr>
<td>ŝ</td>
<td>2.887[298] (0.345)</td>
<td>2.918[317] (0.277)</td>
<td>3.024[405] (0.150)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>s</th>
<th>Left tail</th>
<th>Right tail</th>
<th>Both tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>3.192 (0.638)</td>
<td>4.272 (0.854)</td>
<td>4.445 (0.889)</td>
</tr>
<tr>
<td>50</td>
<td>3.983 (0.563)</td>
<td>3.062 (0.433)</td>
<td>3.917 (0.554)</td>
</tr>
<tr>
<td>75</td>
<td>3.269 (0.373)</td>
<td>3.246 (0.375)</td>
<td>3.672 (0.424)</td>
</tr>
<tr>
<td>100</td>
<td>2.966 (0.297)</td>
<td>3.040 (0.304)</td>
<td>3.554 (0.355)</td>
</tr>
<tr>
<td>320</td>
<td>2.809 (0.157)</td>
<td>2.625 (0.147)</td>
<td>2.925 (0.163)</td>
</tr>
<tr>
<td>ŝ</td>
<td>2.749[335] (0.150)</td>
<td>2.574[365] (0.135)</td>
<td>2.783[474] (0.128)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>s</th>
<th>Left tail</th>
<th>Right tail</th>
<th>Both tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>3.967 (0.793)</td>
<td>4.200 (0.840)</td>
<td>4.461 (0.892)</td>
</tr>
<tr>
<td>50</td>
<td>3.672 (0.519)</td>
<td>3.559 (0.503)</td>
<td>4.269 (0.604)</td>
</tr>
<tr>
<td>75</td>
<td>3.421 (0.395)</td>
<td>3.547 (0.410)</td>
<td>3.825 (0.442)</td>
</tr>
<tr>
<td>100</td>
<td>3.046 (0.305)</td>
<td>3.477 (0.348)</td>
<td>3.615 (0.362)</td>
</tr>
<tr>
<td>200</td>
<td>2.673 (0.189)</td>
<td>2.971 (0.210)</td>
<td>3.107 (0.220)</td>
</tr>
<tr>
<td>ŝ</td>
<td>2.716[161] (0.214)</td>
<td>2.867[161] (0.226)</td>
<td>3.079[204] (0.216)</td>
</tr>
</tbody>
</table>

Note: ŝ: optimal estimate of s using σ = 0.6 and τ = 0.9.
Actual value of ŝ reported in [ ] in each column. Standard errors are shown in parentheses.
Figure 5.2 Tail shapes of return distributions
Figure 5.2 (cont.)
• In these coordinates the Pareto distribution \( P(X < -x) = Dx^{-\zeta} \) appears as a straight line with slope \(-\zeta\). Straight lines of slopes \(-2\) and \(-4\) are graphed against the empirical tails to facilitate comparison.

• These findings again make it clear that the return distribution are certainly fat tailed, but there appears to be little support for them following a stable distribution and thus having an infinite variance.

• Stability of the tail indices was examined by splitting the sample periods in half and computing the \( V_\zeta \) statistics. These results are shown in Table 5.3 along with subsample estimates of the tail indices.

• Only for the right tail of the S&P 500 distribution is there strong evidence of non-constancy. For FT30 and exchange rate distributions, the estimated tail indices seems to be very similar, and certainly there is no evidence of the subperiod estimates close to 2.

• Estimates of extreme levels are shown in Table 5.4.

• For example, the probability that within a given year the FT30 will experience a one-day fall of more than 20% is 0.009, i.e. about once in every 110 years but for the S&P 500 this probability is 0.02, about once every 50 years.

• On the other hand, the probability of the dollar/sterling
exchange rate experience a one-day fall of 20% within a given year, is 0.00325 - once every 300 years!

Table 5.3. *Tail index stability tests*

<table>
<thead>
<tr>
<th></th>
<th>First half</th>
<th></th>
<th>Second half</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\zeta}$</td>
<td>$\hat{s}$</td>
<td>$\hat{\zeta}$</td>
<td>$\hat{s}$</td>
<td>$V_\zeta$</td>
</tr>
<tr>
<td>FT30 left</td>
<td>2.78</td>
<td>200</td>
<td>2.99</td>
<td>201</td>
<td>0.53</td>
</tr>
<tr>
<td>right</td>
<td>2.97</td>
<td>200</td>
<td>3.09</td>
<td>203</td>
<td>0.16</td>
</tr>
<tr>
<td>S&amp;P500 left</td>
<td>3.09</td>
<td>207</td>
<td>3.35</td>
<td>208</td>
<td>0.64</td>
</tr>
<tr>
<td>right</td>
<td>2.48</td>
<td>236</td>
<td>3.48</td>
<td>219</td>
<td>12.41</td>
</tr>
<tr>
<td>Dollar/sterling left</td>
<td>2.37</td>
<td>105</td>
<td>3.07</td>
<td>100</td>
<td>3.37</td>
</tr>
<tr>
<td>right</td>
<td>2.75</td>
<td>100</td>
<td>3.18</td>
<td>100</td>
<td>0.73</td>
</tr>
</tbody>
</table>
Table 5.4. *Lower-tail probabilities*

<table>
<thead>
<tr>
<th>Return</th>
<th>FT30 Probability</th>
<th>Return</th>
<th>S&amp;P 500 Probability</th>
<th>Dollar/stirling Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.200</td>
<td>0.00906</td>
<td>-0.193</td>
<td>0.0100</td>
<td>0.00325</td>
</tr>
<tr>
<td>-0.300</td>
<td>0.00280</td>
<td>-0.246</td>
<td>0.0050</td>
<td>0.00109</td>
</tr>
<tr>
<td>-0.400</td>
<td>0.00122</td>
<td>-0.428</td>
<td>0.0010</td>
<td>0.00050</td>
</tr>
<tr>
<td>-0.500</td>
<td>0.00064</td>
<td>-0.950</td>
<td>0.0001</td>
<td>0.00027</td>
</tr>
</tbody>
</table>

*Note:* Calculated using $k = 260$, $\hat{s} = 298$, $\hat{\zeta} = 2.887$.

<table>
<thead>
<tr>
<th>Return</th>
<th>FT30 Probability</th>
<th>Return</th>
<th>S&amp;P 500 Probability</th>
<th>Dollar/stirling Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.200</td>
<td>0.02019</td>
<td>-0.258</td>
<td>0.0100</td>
<td>0.00302</td>
</tr>
<tr>
<td>-0.300</td>
<td>0.00665</td>
<td>-0.333</td>
<td>0.0050</td>
<td>0.00302</td>
</tr>
<tr>
<td>-0.400</td>
<td>0.00302</td>
<td>-0.598</td>
<td>0.0010</td>
<td>0.00050</td>
</tr>
<tr>
<td>-0.500</td>
<td>0.00164</td>
<td>-1.383</td>
<td>0.0001</td>
<td>0.00027</td>
</tr>
</tbody>
</table>

*Note:* Calculated using $k = 260$, $\hat{s} = 335$, $\hat{\zeta} = 2.749$.

<table>
<thead>
<tr>
<th>Return</th>
<th>FT30 Probability</th>
<th>Return</th>
<th>S&amp;P 500 Probability</th>
<th>Dollar/stirling Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.200</td>
<td>0.00325</td>
<td>-0.132</td>
<td>0.0100</td>
<td>0.00027</td>
</tr>
<tr>
<td>-0.300</td>
<td>0.00109</td>
<td>-0.171</td>
<td>0.0050</td>
<td>0.00027</td>
</tr>
<tr>
<td>-0.400</td>
<td>0.00050</td>
<td>-0.310</td>
<td>0.0010</td>
<td>0.00027</td>
</tr>
<tr>
<td>-0.500</td>
<td>0.00027</td>
<td>-0.725</td>
<td>0.0001</td>
<td>0.00027</td>
</tr>
</tbody>
</table>

*Note:* Calculated using $k = 260$, $\hat{s} = 161$, $\hat{\zeta} = 2.716$. 
Testing for Covariance Stationarity

- The covariance stationarity, that the unconditional variance and covariance do not depend on time, is central to much of time series econometrics.

- In financial market, information and technology are subject to temporal evolution and can be hypothesized to affect the unconditional variance of assets.

- Some conditions on the GARCH process can assure covariance stationarity. For example, in ARCH(1) process, the condition is $\beta < 1$. For GARCH(1, 1), it is $\alpha_1 + \beta_1 < 1$.

- Mandelbrot (1963) proposed the use of $\mu_{2,t} = t^{-1} \sum_{j=1}^{t} X_j^2$ to examine covariance stationarity, which seems to be reasonable if $\hat{\mu}_{2,t}$ converges to a constant.

- The preceding proposal assumes that the maintained distribution is normal, which is obviously inappropriate when dealing with series of returns. Thus Pagan and Schwert (1990) suggest using

$$\psi(r) = (T \hat{\nu})^{-1/2} \left[ \text{Tr} \sum_{j=1}^{[Tr]} (X_j^2 - \hat{\mu}_{2,T}) \right],$$

where $0 < r < 1$ and $\hat{\nu} = \hat{\gamma}_0 + 2 \sum_{j=1}^{l} (1 - \frac{j}{t+1}) \hat{\gamma}_j$ is a kernel-based estimate of the ‘long-run’ variance of $X_i^2$, using the covariance $\hat{\gamma}_0, \cdots, \hat{\gamma}_l$ of the series.
• This is a studentized version of the cumulative sum of squares statistics.

• Inference about $\psi(r)$ depends crucially on the value taken by the tail index $\zeta$ of the distribution of $X$.

• For $\zeta > 4$, and $T \to \infty$, $\psi(r)$ converges weakly to a Brownian bridge. Thus the probability that $\psi(r) < c$ equals to the probability that a $N(0, r(1-r))$ random variable less than $c$.

• For $\zeta < 4$, $\psi(r)$ converges to a standardized tied-down stable process. Critical values thus depend in a complicated fashion on $\zeta$ and are tabulated in Loretan and Philips (1994, table 2).

• For $\zeta > 4$, the 5% critical value of $\psi(0.9)$ is 0.49, whereas for $\zeta = 2.1$, it is 0.27. However, while $\zeta > 4$ 5% critical value of $\psi(0.1)$ is also 0.49 because of the symmetry of the limit distribution, for $\zeta = 2.1$, it is 0.66.

• The test has decreasing power as $\zeta$ tends to 2 from above as the rate of divergence from the null become slower due to the presence of increasing amount of outliers. For $\zeta \leq 2$, the test is inconsistent as in this case variances are infinite.

• We can also consider the range statistic $R = \sup_r \psi(r) - \inf_r \psi(r)$, which is identical to the Lo’s (1991) rescaled range statistic discussed in ARFIMA models.
Modeling the Central Part of Returns Distribution

- Let
  \[ X_t(k) = \sum_{i=0}^{k} X_{t-i} \]
denote the \( k \)-period non-overlapping return, for different values of \( k \).

- Mantegna and Stanley (1995) consider fitting stable distribution to the central part of a return distribution utilizing the ‘probability of return’ \( P(X_t(k) = 0) \).

- For stable distribution, we have
  \[ P(X_t(k) = 0) = \frac{\gamma(1/\alpha)}{\pi\alpha(\gamma k)^{1/\alpha}}. \]

- We can estimate \( \alpha \) as minus the inverse of the slope in a logarithmic regression of \( P(X_t(k) = 0) \) on \( k \).

- This probability may be estimated as the frequency of \(-x < X_t(k) \leq x\), where the value of \( x \) is chosen to be a suitably small number depending on the scale of the observed returns and the range of \( k \) values is chosen to reflect the length and frequency of the series.

- Using transaction data on S&P 500 for the six-year period from 1984 to 1989 and obtained an estimate of 1.40. With daily data over 64 years, we obtain \( \alpha = 1.42 \): a remarkable confirmation of the invariance property of stable distribution.
• Mangegna and Stanley remark on the tightness of the stable fit for values within about six standard deviation of zero, outside of which the tails of the distribution seem to follow an exponential, rather than a stable, decline.

• This is consistent with the finding that the central part of a return distribution seems to follow a stable law, while the tails undergo an exponential decline so that the variance of the distribution is finite.
Data Analytic Modeling of skewness and kurtosis

- Skewness is important both because of its impact on portfolio choice and because kurtosis is not independent of skewness.

- The skewness reported in Table 5.1 are all negative and significantly different from zero on using the fact that

\[ \sqrt{\frac{T}{6}} \cdot \text{skew} \sim N(0, 1). \]

- The median is defined as \( X_{med} = X([T/2]) \). For symmetric distribution, the order statistics \( X(p), X(T-p), p < [T/2] \), are equidistant from the median,

\[ X(T-p) - X_{med} = X_{med} - X(p). \]

- The plot of the upper-order statistic \( X(T-p) \) against the lower order statistic \( X(p) \) should be linear with a slope of \(-1\) if the distribution is symmetric.

- Figure 5.4 suggests that the distribution are symmetric over a wide range of values, with asymmetry only appearing in the tails of the distributions.

- The asymmetry is characterized by negative skewness, so that there is greater probability of large falls in price than large increases.
Table 5.1. Descriptive statistics on returns distributions

<table>
<thead>
<tr>
<th></th>
<th>$T$</th>
<th>mean</th>
<th>median</th>
<th>std.dev</th>
<th>max</th>
<th>min</th>
<th>range</th>
<th>skew</th>
<th>kurt</th>
</tr>
</thead>
<tbody>
<tr>
<td>FT30</td>
<td>15003</td>
<td>0.022</td>
<td>0.000</td>
<td>1.004</td>
<td>10.78</td>
<td>-12.40</td>
<td>23.1</td>
<td>-0.14</td>
<td>14.53</td>
</tr>
<tr>
<td>Dollar</td>
<td>5192</td>
<td>-0.008</td>
<td>0.000</td>
<td>0.647</td>
<td>4.67</td>
<td>-3.87</td>
<td>13.2</td>
<td>-0.70</td>
<td>6.51</td>
</tr>
</tbody>
</table>
Figure 5.4 ‘Upper-lower’ symmetry plots
Figure 5.4 (cont.)
Distributional Properties of Absolute Returns

- Let $X_t = |X_t| \cdot X_t$, where

  $$\text{sign } X_t = \begin{cases} 
  1 & \text{if } X_t > 0 \\
  0 & \text{if } X_t = 0 \\
  -1 & \text{if } X_t < 0 
  \end{cases}$$

- Granger and Ding (1995) suggest three distributional properties related to the preceding decomposition.
  
  - $|X_t|$ and sign $X_t$ are independent.
  
  - The mean and variance of $|X_t|$ are equal.
  
  - The marginal distribution of $|X_t|$ is exponential after outlier reduction.

- Note that an exponential distribution with parameter $\lambda$ has both mean and variance equal to $\lambda$, a skewness of 2 and kurtosis of 9.

- Grander and Ding show that all three properties hold for S&P 500 series, which is confirmed in Table 5.7

- The outlier reduction data in Table 5.7 is produced by replacing any observation greater than four times the standard deviation(S.D.) by 4S.D. value having the same sign.

- For the two stock returns series there is evidence of asymmetry, while the evidence in the exchange rate is much less.
• The estimated conditional means and standard deviation of the outlier adjusted series are approximately equal, and the skewness and kurtosis are close to 2 and 9 respectively.

• For all series, that $|X_t|$ and sign are independent, is confirmed using a Kolmogorov-Smirnov test.

• It is also argued that, if $|X_t|$ is exponential, then it is reasonable to expect that the pair $|X_t|, |X_{t-k}|$ will be jointly exponential.

• This joint distribution has the properties that the marginal distributions are each exponential and that the conditional mean $E(|X_t| | |X_{t-k}|)$ is a linear function of $|E(X_{t-k})|$.

• This suggests that linear regression of absolute returns on lagged absolute returns may have some predictive power. However, empirical studies reveal that the predictive power is rather weak.
Table 5.7. Properties of marginal return distributions.

<table>
<thead>
<tr>
<th></th>
<th>Observed +</th>
<th>Observed -</th>
<th>outlier adjusted +</th>
<th>outlier adjusted -</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FT50</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prob</td>
<td>0.50</td>
<td>0.44</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Mean × 100</td>
<td>0.69</td>
<td>0.73</td>
<td>0.68</td>
<td>0.71</td>
</tr>
<tr>
<td>S.D. × 100</td>
<td>0.74</td>
<td>0.78</td>
<td>0.66</td>
<td>0.70</td>
</tr>
<tr>
<td>Mean/S.D.</td>
<td>0.93</td>
<td>0.93</td>
<td>1.02</td>
<td>1.01</td>
</tr>
<tr>
<td>Skewness</td>
<td>3.55</td>
<td>3.55</td>
<td>2.10</td>
<td>2.01</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>26.87</td>
<td>28.76</td>
<td>8.77</td>
<td>8.19</td>
</tr>
<tr>
<td>outliers</td>
<td>-</td>
<td>-</td>
<td>95</td>
<td>50</td>
</tr>
<tr>
<td><strong>S&amp;P 500</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prob</td>
<td>0.52</td>
<td>0.46</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Mean × 100</td>
<td>0.72</td>
<td>0.73</td>
<td>0.71</td>
<td>0.76</td>
</tr>
<tr>
<td>S.D. × 100</td>
<td>0.85</td>
<td>0.94</td>
<td>0.74</td>
<td>0.81</td>
</tr>
<tr>
<td>Mean/S.D.</td>
<td>0.86</td>
<td>0.82</td>
<td>0.96</td>
<td>0.93</td>
</tr>
<tr>
<td>Skewness</td>
<td>4.30</td>
<td>4.82</td>
<td>2.49</td>
<td>2.30</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>37.97</td>
<td>59.13</td>
<td>11.13</td>
<td>9.45</td>
</tr>
<tr>
<td>outliers</td>
<td>-</td>
<td>-</td>
<td>138</td>
<td>74</td>
</tr>
<tr>
<td><strong>Dollar/Stirling</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prob</td>
<td>0.46</td>
<td>0.45</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Mean × 100</td>
<td>0.35</td>
<td>0.40</td>
<td>0.35</td>
<td>0.39</td>
</tr>
<tr>
<td>S.D. × 100</td>
<td>0.32</td>
<td>0.40</td>
<td>0.31</td>
<td>0.37</td>
</tr>
<tr>
<td>Mean/S.D.</td>
<td>1.10</td>
<td>0.99</td>
<td>1.14</td>
<td>1.06</td>
</tr>
<tr>
<td>Skewness</td>
<td>2.29</td>
<td>1.94</td>
<td>2.14</td>
<td>1.96</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>12.29</td>
<td>8.21</td>
<td>9.24</td>
<td>7.46</td>
</tr>
<tr>
<td>outliers</td>
<td>-</td>
<td>-</td>
<td>36</td>
<td>23</td>
</tr>
</tbody>
</table>